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Induced Saturation Number

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Induced saturation number

by

Jason James Smith

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:

Ryan Martin, Major Professor

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Ling Long

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Ames, Iowa

2012

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ABSTRACT

In this paper, we discuss the induced saturation number. It is a nice generalization of the saturation number that will allow us to consider induced subgraphs. We define the induced saturation number, $\text{indsat}(n, H)$, to be the fewest number of gray edges in a trigraph T such that H does not appear in any realization of T , but if a black or white edge of T is flipped to gray then there exists a realization of T with H as an induced subgraph. We will provide some general results and prove that for a path on four vertices, $\text{indsat}(n, P_4) = \left\lceil \frac{n+1}{3} \right\rceil$ for $n \geq 4$. We will also discuss the injective coloring number and a generalization of that.

CHAPTER 1. INTRODUCTION

In this section we will provide a brief review of the concepts already published related to our work, we will then define terms and notations related to our work, and finally we will provide a brief justification for the work.

1.1 Saturation in Graphs

We start with the definitions laid out by Kászonyi and Tuza in [23]. We say a graph G is H -saturated if it does not contain H as a subgraph, but H occurs whenever any new edge is added to G . The Turán type problems deal with H -saturated graphs, in particular $ex(n; H) = \max\{|E(G)| : |V(G)| = n, G \text{ is } H\text{-saturated}\}$. Although previous work had been done in the field, Kászonyi and Tuza were the first to formally define the saturation number to be $\text{sat}(n, H) = \min\{|E(G)| : |V(G)| = n, G \text{ is } H\text{-saturated}\}$. Kászonyi and Tuza find the saturation number for paths, stars, and matchings. In particular they find

$$\text{sat}(n, P_3) = \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{sat}(n, P_4) = \begin{cases} k, & \text{if } n = 2k; \\ k + 1, & \text{if } n = 2k - 1. \end{cases}, \quad \text{sat}(n, P_5) = n - \left(\left\lfloor \frac{n-2}{6} \right\rfloor + 1 \right),$$

$\text{sat}(n, P_m) = n - \left\lfloor \frac{n}{3 \cdot 2^{k-1} - 1} \right\rfloor$ when $m = 2k \geq 6$ and $\text{sat}(n, P_m) = n - \left\lfloor \frac{n}{2^{k+1} - 2} \right\rfloor$ when $m = 2k + 1 \geq 7$. They also find a general upper bound for saturation number. That is for any graph H there exists a constant $c(H)$, such that $\text{sat}(n, H) < cn$.

Pre-dating [23], Erdős, Hajnal, and Moon found the saturation number of complete graphs in [15], that is $\text{sat}(n, K_p) = (p-2)n - \binom{p-1}{2}$. In 1972, Ollmann in [27] found the saturation number of the four cycle to be $\lceil 3(n-5)/2 \rceil$. More recently, Faudree et al. in [18] found the saturation number of tK_p , Chen in [7] found the saturation number of C_5 , and Chen in [6] found the saturation number of $K_{2,3}$. A more complete background of known saturation results is

provided in the dynamic survey by Faudree, Faudree, and Schmitt in [16]. Of particular interest are the papers in [1], [3], [4], [5], [8], [13], [14], [17], [19], [20], [21], [29].

1.1.1 Definitions and Notation

We start with the basics. A *graph* G is an ordered pair (V, E) , where $V = \{v_1, v_2, \dots, v_k\}$ is the set of vertices and $E \subseteq \{\{v_i, v_j\} : 1 \leq i < j \leq k\}$ is the set of edges. All of our graphs will be simple, undirected, and finite. A *subgraph* of a graph G is a graph whose vertex set is a subset of $V(G)$, and whose adjacency relation is a subset of $E(G)$ restricted to this subset. A subgraph H of a graph G is an *induced subgraph* if for any pair of vertices x and y of H , xy is an edge of H if and only if xy is an edge of G , in other words H is the subgraph induced from G by $V(H)$.

In order to generalize the idea of saturation, we will use the definitions given by Chudnovsky in [9]. A *trigraph* is a quadruple $(V(T); EB(T), EW(T), EG(T))$ in which $(EB(T), EW(T), EG(T))$ is a partition of the edges of the complete graph on the vertex set $V(T)$. (Possibly one or two of $EB(T), EW(T), EG(T)$ can be the empty set.) We call the set $EB(T)$ the *black edges* of T , the set $EW(T)$ the *white edges* of T , and the set $EG(T)$ the *gray edges* of T . We will think of these as edges, non-edges, and ‘free’ edges, where ‘free’ means that we may regard the gray edges as either black or white. We note here that if $EG(T) = \emptyset$, then our trigraph may be regarded as just a graph, in which the black edges are treated as ‘edges’ and the white edges are treated as ‘nonedges’. We say a *realization* of a trigraph T is a graph G with $V(G) = V(T)$ and $E(G) = EB(T) \cup S$ for some subset S of $EG(T)$. That is, we may see this as setting some gray edges to be black and setting the remaining gray edges to white. We say the complement, \bar{T} of a trigraph T is a trigraph with $V(\bar{T}) = V(T)$, $EB(\bar{T}) = EW(T)$, $EG(\bar{T}) = EG(T)$, and $EW(\bar{T}) = EB(T)$, which we may view as extending the definition of graph complement.

With the above definitions, we start the definition of the induced saturation number. For any graph H , we say that a trigraph T *has a realization of H* if there is a realization of T which has H as an induced subgraph. A trigraph T is *H -induced-saturated* if no realization of T contains H as an induced subgraph, but H occurs as an induced subgraph of realization whenever

any black or white edge of T is changed to gray. The induced saturation number of H with respect to n is defined to be $\text{indsat}(n, H) = \min\{|EG(T)| : |V(T)| = n, T \text{ is } H\text{-induced-saturated}\}$.

In plain English, to find the induced saturation number we need to find a trigraph T on n vertices with the fewest number of gray edges such that T does not contain H as an induced subgraph, but if we flip any white or black edge of T to gray, then we find H as an induced subgraph.

We note that with this definition, the only trigraphs on fewer than $|V(H)|$ vertices that are H -induced-saturated are those which are complete graphs of gray edges.

In an attempt to follow the ‘+’ notation used in saturation, we denote $T \div e$ as the process of taking the trigraph T and changing the otherwise nongray edge, e , to be a gray edge. This notation makes sense as we are either adding or subtracting that edge e . In another extension of saturation terminology, we will say that an edge e is not *induced – critical*, to mean that $T \div e$ does not contain the desired induced subgraph.

If an edge has end vertices v and w , we denote it vw or wv . For a vertex v in a trigraph T , the *white neighborhood* of v is the set $\{w : vw \in EW(T)\}$. The *black neighborhood* and *gray neighborhood* are defined similarly. We define a *white component* to be a set of vertices that are connected in the graph $(V(T), EW(T))$. We do not require that every edge is white, just that there is a white path connecting the vertices. In a similar fashion, we define *gray*, *black*, *black/gray*, and *white/gray components*.

If V_1 and V_2 are disjoint sets in $V(T)$, then $T[V_1, V_2]$ denotes the set of edges with one endpoint in V_1 and the other endpoint in V_2 . Further $T[V_1]$ denotes the subtrigraph induced by V_1 .

In a graph, we say that vertices $v_1 v_2 \dots v_k$ form a path if $v_i v_{i+1}$ is an edge for $i = 1, \dots, k-1$, but all other pairs of these vertices are nonedges, we will denote this P_k . Likewise, we say that vertices $v_1 v_2 \dots v_k$ form a star if $v_1 v_i$ is an edge for $i = 2, \dots, k$, but all other pairs of these vertices are nonedges, we will denote this S_k . Further, we say that vertices $v_1 v_2 \dots v_k$ form a complete graph if $v_i v_j$ is an edge for $1 \leq i < j \leq k$, and there are no nonedges, we will denote this K_k .

1.1.2 Applications

In the general setting, induced saturation is a satisfiability problem. One is given a disjunctive normal form (DNF) and wants to find a partial assignment of variables such that (1) there is no way to complete the assignment to a true one but (2) if any of the variables were unassigned, then the partial assignment can be completed to a true one.

In the specific case of the graph P_4 , the DNF constructed from the set of pairs of n vertices is one comprised of clauses, each of which represents an instance of a potential induced copy of P_4 . For instance, if e_1, \dots, e_6 represent unordered pairs such that e_1, e_2, e_3 being edges and e_4, e_5, e_6 being nonedges induces a P_4 , then the corresponding clause is $x_1 \wedge x_2 \wedge x_3 \wedge \overline{x_4} \wedge \overline{x_5} \wedge \overline{x_6}$, where the variable x_i corresponds to the edge e_i , for $i = 1, \dots, 6$.

The trigraph has a number of applications related to Szemerédi's regularity lemma [28] (see also [24], [25]). A trigraph can also be thought of as a reduced graph in which a black edge represents a pair with density close to 1, a white edge represents a pair with density close to 0, and a gray edge represents a pair with density neither near 0 nor 1. Such a configuration is used in a number of applications of the regularity lemma related to induced subgraphs. For instance, see [2].

CHAPTER 2. INDUCED SATURATION NUMBER AND BOUNDS

This chapter will provide the induced saturation number of a few families of graphs as well as provide bounds for the induced saturation number. These results will show that the induced saturation number does not always behave as one might expect.

For definitions of a trigraph, realization, trigraph complement, H -induced-saturated and the induced saturation number, see Section 1.1.1. The majority of the results in this chapter were the initial results found about the induced saturation number. We were focusing on showing that the induced saturation number is different than the saturation number.

In the first section of this chapter, we will provide a bound for the induced saturation number using the saturation number. In addition we will provide a bound on the induced saturation number of a path of any length. In the second section of this chapter, we will establish the induced saturation number for a few families of graphs.

2.1 Bounds for Induced Saturation Number

In this section we will provide a general bound on the induced saturation number and then provide a bound on the induced saturation number of paths of any length.

Our first result nicely relates the saturation number to the induced saturation number. When this result is coupled with the general bound for saturation found by Kászonyi and Tuza [23], we get a general bound on the induced saturation number.

Theorem 2.1.1. *$indsat(n, H) \leq sat(n, H)$ for all $n \in \mathbb{N}$ and all graphs H .*

Proof. Let H be a graph and G be a graph which witnesses the saturation number for H , that is G is H -saturated and has $sat(n, H)$ edges. From G we construct a trigraph T by changing the edges of G to gray edges in T and changing the non-edges of G to white edges in T . It is

straight-forward to see that $|EG(T)| = \text{sat}(n, H)$. Thus, we need to show that T is H -induced-saturated. First, we note that H does not appear as a subgraph in T , by the way T was constructed from G . Second let e be a white edge in T . We consider $T \dot{\vdash} e$, we must have H appearing as a gray subgraph in $T \dot{\vdash} e$ again by the way T was constructed from G and the fact that G is H -saturated. Therefore, our proof is complete. \square

Now we have a result for general paths. This shows that for an entire family of graphs the induced saturation number can be strictly less than the saturation number. In fact, the result below shows that for general paths $\text{indsat}(n, P_m) \leq 0.75n$ whereas $\text{sat}(n, P_m) \approx n$, by which we mean m is arbitrarily large but fixed and n is growing.

Theorem 2.1.2. *Let P_m be a path on m vertices.*

$\text{indsat}(n, P_m) \leq [9(2^{k-3}) - 3]p$, for $m = 2k$, $n = [3(2^{k-1}) - 2]p$ with $m \geq 3$, $p \geq 1$.

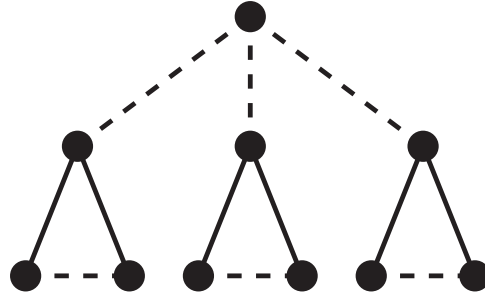
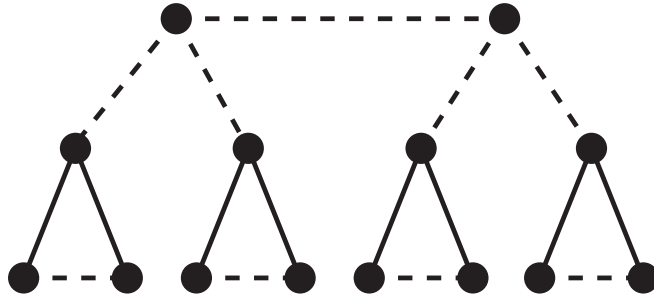
$\text{indsat}(n, P_m) \leq [6(2^{k-2}) - 3]p$, for $m = 2k + 1$, $n = [2^{k+1} - 2]p$ with $m \geq 3$, $p \geq 1$.

Proof. It suffices to provide a trigraph T which is P_m -induced-saturated and has the appropriate number of gray edges. To this end, we define a trigraph T_m , which looks like an almost binary, gray tree. We will use tree terminology to describe T_m . T_m has $\lfloor \frac{m}{2} \rfloor$ levels in which every vertex has degree 3, except for the lowest level. The highest level contains $m+1-2\lfloor \frac{m}{2} \rfloor$ vertices. Further leaves which are siblings are connected by a gray edge, and all edges connecting leaves to their parents are black as in Figure 2.1.

We will construct our trigraph T by taking the disjoint union of p copies of T_m . Considering this, we only need to count the number of gray edges in T_m . We will show the counting for the case when $m = 2k$. A similar argument works for the case when $m = 2k + 1$.

Let $m = 2k$. By inspection, we can see that the number of gray edges in T_m is $|EG(T_m)| = (3 + 6 + 12 + \dots + 3 \cdot 2^{k-3}) + 3 \cdot 2^{k-3}$, where the count inside the parenthesis is the gray edges in the upper 2^{k-1} levels of the tree and the second term is the number of gray edges between the leaves. Now we factor out a 3 and calculate the finite sum of the resulting geometric series, and finally combine like terms. That is,

$$|EG(T_m)| = 3(1 + 2 + 2^2 + \dots + 2^{k-3}) + 3 \cdot 2^{k-3} = 3(2^{k-2} - 1) + 3 \cdot 2^{k-3} = 9(2^{k-3}) - 3.$$

(a) T_6 (b) T_7 Figure 2.1: Examples of T_m , for $m = 6, 7$.

Now we must show that T_m is P_m -induced-saturated. We have multiple cases to consider. We will assign ordered pairs to each vertex in our graph T_m , the first coordinate will be its depth in the tree (root(s) have depth 0). The second coordinate will be its horizontal position, again starting at 0. The coordinate system is shown in Figure 2.2.

In all cases we'll assume we are flipping the edge, $\{(a_1, b_1), (a_2, b_2)\}$. After flipping this edge we will demonstrate how to find an induced path of length $m - 1$. In doing this, we will use the notation $(r_1, s_1) \rightarrow (r_2, s_2)$ to mean the direct path from (r_1, s_1) to (r_2, s_2) when it is clear that such a path exists. We will also use partially ordered set notation, that is $a \vee b$ is the join of a and b , with the join meaning the nearest common ancestor if such an ancestor exists. Also, in all cases we will leave it to the interested reader to verify that the path is induced.

Case 1 : $\{m = 2k\}$. In this case m is even. As such we have a single root, $(0, 0)$.

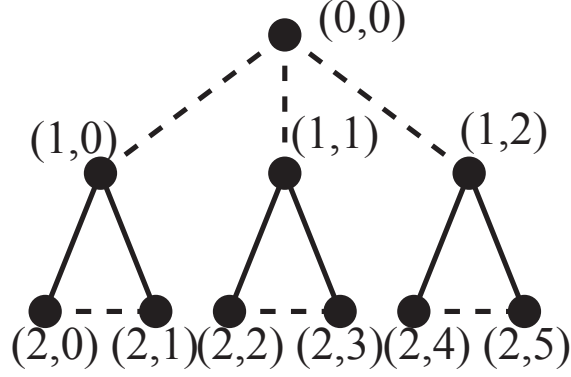


Figure 2.2: T_m with coordinate system.

Subcase 1a : $\{a_2 \geq a_1 \text{ and } (a_1, b_1) \vee (a_2, b_2) \neq (a_1, b_1)\}$. In this case the two vertices are not related and (a_1, b_1) is closer to the root. As $\{(a_1, b_1), (a_2, b_2)\}$ was originally a white edge, we will use it as a black edge. We choose a leaf which is a descendent of (a_1, b_1) , say (r_1, s_1) . Now the path, P , we consider is

$$(r_1, s_1) \rightarrow (a_1, b_1), (a_2, b_2) \rightarrow (0, 0) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is a leaf which is neither a descendent of (a_1, b_1) nor a descendent of (a_2, b_2) . Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_1, b_1) is $(k-1)-a_1$ since (r_1, s_1) is a leaf. The length from (a_1, b_1) to (a_2, b_2) is 1. The length from (a_2, b_2) to $(0, 0)$ is a_2 . The length from $(0, 0)$ to (r_2, s_2) is $k-1$. Therefore, we have a total length of $k-1-a_1+1+a_2+k-1$, which is $2k-1+a_2-a_1$. Now, since we are in the case with $a_2 \geq a_1$ we must have that the length of P is at least $2k-1$. That is to say, P is a path on at least $2k$ or m vertices.

Subcase 1b : $\{a_2 = a_1 + 1 \text{ and } (a_1, b_1) \vee (a_2, b_2) = (a_1, b_1)\}$. In this case (a_1, b_1) is the parent of (a_2, b_2) . We note that it must be that (a_2, b_2) is a leaf, since all other edges are already gray. We choose (a_2, b_2) 's sibling, say (r_1, s_1) . Now the path, P , we consider is

$$(r_1, s_1), (a_2, b_2), (a_1, b_1) \rightarrow (0, 0) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is not a descendent of (a_1, b_1) . Now we will calculate the length of P , by cal-

culating the lengths of each segment. The length from (r_1, s_1) to (a_2, b_2) is 1. The length from (a_2, b_2) to (a_1, b_2) is 1. The length from (a_2, b_2) to $(0, 0)$ is $k - 2$, since (a_2, b_2) is the parent of a leaf. The length from $(0, 0)$ to (r_2, s_2) is $k - 1$. Therefore, we have a total length of $1 + 1 + k - 2 + k - 1$, which is $2k - 1$. Thus, we have that the length of P is $2k - 1$. That is to say, P is a path on $2k$ or m vertices.

Subcase 1c : $\{a_2 \geq a_1 + 2 \text{ and } (a_1, b_1) \vee (a_2, b_2) = (a_1, b_1)\}$. In this case (a_2, b_2) is a non-child descendent of (a_1, b_1) . As $\{(a_1, b_1), (a_2, b_2)\}$ was originally a white edge, we will use it as a black edge. We note that we are not excluding the case when (a_1, b_1) is the root. We start by noting that there exists a path from (a_2, b_2) to a leaf (r_1, s_1) with $a_2 - a_1 - 1 + k - a_1 - 2$ edges that lies in the subtree generated by the child of (a_1, b_1) which is an ancestor of (a_2, b_2) (see Figure 2.3). In the figure, the finely dashed line is the new added edge. Now the path, P , we consider is

$$(r_1, s_1) \rightarrow (a_2, b_2), (a_1, b_1) \rightarrow (0, 0) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is a leaf which not a descendent of (a_2, b_2) , and is not a descendent of (a_1, b_1) if (a_1, b_1) is not the root. Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_2, b_2) is $a_2 - a_1 - 1 + k - a_1 - 2$. The length from (a_2, b_2) to (a_1, b_2) is 1. The length from (a_2, b_2) to $(0, 0)$ is a_1 . The length from $(0, 0)$ to (r_2, s_2) is $k - 1$. Therefore, we have a total length of $a_2 - a_1 - 1 + k - a_1 - 2 + 1 + a_1 + k - 1$, which is $2k - 3 + a_2 - a_1$. Now, since in this case we have that $a_2 - a_1 \geq 2$, it must be that the length of P is at least $2k - 1$. That is to say, P is a path on $2k$ or m vertices.

Therefore we have exhausted all cases for m being even. Next we consider the cases when m is odd.

Case 2 : $\{m = 2k + 1\}$. In this case m is odd. As such we have two roots, say $(0, 0)$ and $(0, 1)$. Unless otherwise stated we'll assume everything is happening in the subtree generated by $(0, 0)$ as symmetry will take care similar situations in the subtree generated by $(0, 1)$.

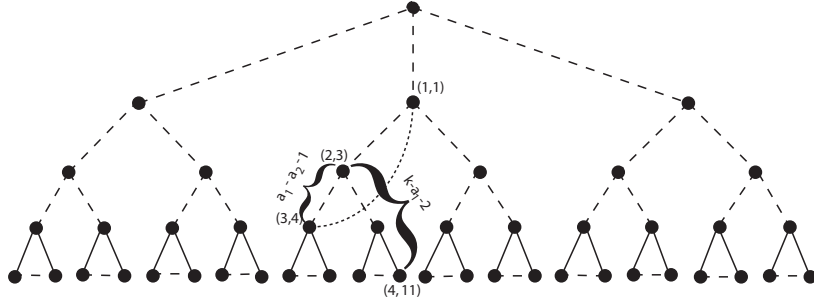


Figure 2.3: The gray edge added in Case 1c.

Case 2a : $\{a_2 \geq a_1 \text{ and } (a_1, b_1) \vee (a_2, b_2) \neq (a_1, b_1) \text{ with } (a_1, b_1) \text{ a proper descendent of } (0,0)\}$. In this case the two vertices are not related and (a_1, b_1) is closer to the root, but is not actually the root. As $\{(a_1, b_1), (a_2, b_2)\}$ was originally a white edge, we will use it as a black edge. We choose a leaf which is a descendent of (a_1, b_1) , say (r_1, s_1) . Now the path, P , we consider is

$$(r_1, s_1) \rightarrow (a_1, b_1), (a_2, b_2) \rightarrow (0,0), (0,1) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is a leaf which is neither a descendent of (a_1, b_1) nor a descendent of (a_2, b_2) . Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_1, b_1) is $(k-1) - a_1$ since (r_1, s_1) is a leaf. The length from (a_1, b_1) to (a_2, b_2) is 1. The length from (a_2, b_2) to $(0,0)$ is a_2 . The length from $(0,0)$ to $(0,1)$ is 1. The length from $(0,1)$ to (r_2, s_2) is $k-1$. Therefore, we have a total length of $k-1 - a_1 + 1 + a_2 = 1 + k - 1$, which is $2k + a_2 - a_1$. Now, since we are in the case with $a_2 \geq a_1$ we must have that the length of P is at least $2k$. That is to say, P is a path on at least $2k + 1$ or m vertices.

Case 2b : $\{a_2 > a_1 \text{ and } (a_1, b_1) \vee (a_2, b_2) \neq (a_1, b_1) \text{ and } (a_1, b_1) \text{ is the root } (0,0)\}$. In this case (a_2, b_2) must be a descendent of $(0,1)$. As $\{(a_1, b_1), (a_2, b_2)\}$ was originally a white edge, we will use it as a black edge. We choose a leaf which is a descendent of (a_1, b_1) , say (r_1, s_1) . Now the path, P , we consider is

$$(r_1, s_1) \rightarrow (a_1, b_1), (a_2, b_2) \rightarrow (0,1) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is a leaf which is neither a descendent of (a_1, b_1) nor a descendent of (a_2, b_2) .

Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_1, b_1) is $k - 1$ since (r_1, s_1) is a leaf and (a_1, b_1) is a root. The length from (a_1, b_1) to (a_2, b_2) is 1. The length from (a_2, b_2) to $(0, 1)$ is a_2 . The length from $(0, 1)$ to (r_2, s_2) is $k - 1$. Therefore, we have a total length of $k - 1 + 1 + a_2 + k - 1$, which is $2k + a_2 - 1$. Now, since we are in the case with $a_2 > a_1 = 0$ we must have that the length of P is at least $2k$. That is to say, P is a path on at least $2k + 1$ or m vertices.

Case 2c : $\{a_2 = a_1 + 1 \text{ and } (a_1, b_1) \vee (a_2, b_2) = (a_1, b_1)\}$. In this case (a_1, b_1) is the parent of (a_2, b_2) . We note that it must be that (a_2, b_2) is a leaf, since all other edges are already gray. We choose (a_2, b_2) 's sibling, say (r_1, s_1) . Now the path, P , we consider is

$$(r_1, s_1), (a_2, b_2), (a_1, b_1) \rightarrow (0, 0), (0, 1) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is not a descendent of (a_1, b_1) . Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_2, b_2) is 1. The length from (a_2, b_2) to (a_1, b_1) is 1. The length from (a_2, b_2) to $(0, 0)$ is $k - 2$, since (a_2, b_2) is the parent of a leaf. The length from $(0, 0)$ to $(0, 1)$ is 1. The length from $(0, 1)$ to (r_2, s_2) is $k - 1$. Therefore, we have a total length of $1 + 1 + k - 2 + 1 + k - 1$, which is $2k$. That is to say, P is a path on $2k + 1$ or m vertices.

Case 2d : $\{a_2 \geq a_1 + 2 \text{ and } (a_1, b_1) \vee (a_2, b_2) = (a_1, b_1)\}$. In this case (a_2, b_2) is a non-child descendent of (a_1, b_1) . As $\{(a_1, b_1), (a_2, b_2)\}$ was originally a white edge, we will use it as a black edge. We note that we are not excluding the case when (a_1, b_1) is the root $(0, 0)$. We start by noting that there exists a path from (a_2, b_2) to a leaf (r_1, s_1) with $a_2 - a_1 - 1 + k - a_1 - 2$ edges that lies in the subtree generated by the child of (a_1, b_1) which is an ancestor of (a_2, b_2) (same idea as Case 1c and Figure 2.3). Now the path, P , we consider is

$$(r_1, s_1) \rightarrow (a_2, b_2), (a_1, b_1) \rightarrow (0, 0), (0, 1) \rightarrow (r_2, s_2),$$

where (r_2, s_2) is a leaf which is a descendent of $(0, 1)$. Now we will calculate the length of P , by calculating the lengths of each segment. The length from (r_1, s_1) to (a_2, b_2) is $a_2 - a_1 - 1 + k - a_1 - 2$.

The length from (a_2, b_2) to (a_1, b_2) is 1. The length from (a_2, b_2) to $(0, 0)$ is a_1 . The length from $(0, 0)$ to $(0, 1)$ is 1. The length from $(0, 0)$ to (r_2, s_2) is $k - 1$. Therefore, we have a total length of $a_2 - a_1 - 1 + k - a_1 - 2 + 1 + a_1 + 1 + k - 1$, which is $2k - 2 + a_2 - a_1$. Now, since in this case we have that $a_2 - a_1 \geq 2$, it must be that the length of P is at least $2k$. That is to say, P is a path on $2k + 1$ or m vertices.

Therefore we have exhausted all possible cases and the proof of Theorem 2.1.2 is complete. \square

The above theorem is very exciting as it shows that the induced saturation number and the saturation number are different numbers for some graphs. In the next section we focus on the induced saturation number and present a few nice results for a couple families of graphs with unexpected induced saturation numbers.

2.2 Induced Saturation Number for a Few Families of Graphs

As mentioned above, our first focus was to show that the induced saturation number is different than the saturation number for at least some graphs. We achieved this goal in the above section. In this section we have results that show that the induced saturation number can range from zero all the way up to the saturation number, which shows the bound given above is tight. The first two theorems provided are surprising in the sense that the two induced subgraphs seem similar but the induced saturation number is very different.

Theorem 2.2.1. $\text{indsat}(n, K_m) = \text{sat}(n, K_m)$ for $n \geq m$.

Proof. From Theorem 2.1.1, we know $\text{indsat}(n, K_m) \leq \text{sat}(n, K_m)$. Thus, we need to show $\text{indsat}(n, K_m) \geq \text{sat}(n, K_m)$. To this end, let H be a complete graph and T be a trigraph that is a witness for $\text{indsat}(n, H)$, that is T is H -induced-saturated and $|EG(T)| = \text{indsat}(n, H)$. We will show that T has no black edges. For sake of contradiction assume T does have a black edge, e . Now we consider a complete graph K in $T \div e$. As e is the edge that we flipped, K must use e , further K must contain e as a white edge since e was originally black. This is the sought contradiction, as K does not have any white edges. Therefore, we know that T has only white and gray edges. Now, we turn T into a graph G , by turning all gray edges to edges of

G and turning all white edges to non-edges of G . We must now show that G is a H -saturated graph. To this end, let e be a non-edge in G and consider $G+e$. To see that $G+e$ contains H as a subgraph, we note that e was also white in the trigraph T , and $T \div e$ contained H as an induced subgraph (completely in gray), so the same vertices that induced H in T will induce H in G . \square

As stated previously, this result shows that the induced saturation number can be the same as the saturation number. The following two results, show that the induced saturation number can get as low as zero.

Theorem 2.2.2. $indsat(n, K_m \setminus e) = 0$, where e is any edge of the K_m for all n .

Proof. We start by finding an upper bound. We let T be the complete graph of black edges on n vertices. To see that T is $K_m \setminus e$ -induced-saturated, we change one of the black edges say e' to gray. Now, we use e' as e , that is let e' be the white edge in $K_m \setminus e$. \square

We have an immediate corollary of the above result.

Corollary 2.2.3. $indsat(n, P_3) = 0$, for all $n \geq 3$.

Proof. This is a direct result of Theorem 2.2.2. \square

Now we present the second result that establishes that induced saturation number is different than saturation number. We will use the notation I_s to mean the graph of s vertices with no edges, that is the independent set on s vertices.

Theorem 2.2.4. $indsat(n, P_3 \cup I_s) = 0$, for all $n \geq 3(s+2)$.

Proof. We start by finding an upper bound. For notational simplicity let $H = P_3 \cup I_s$. We let T be the disjoint union of black K_3 s and one K_m for $m = 4, 5$, in the cases when n is not divisible by 3. We note that H does not appear in T because there is no P_3 in each of the K_3 s. Now, we have two cases to consider. First, let e be a black edge in T . It must be that e was an edge in one of the K_3 s. We consider $T \div e$. We must use e as a white edge, but this gives us a P_3 using the two edges adjacent to e . Further, there are enough other disjoint K_3 s to take one vertex of each and form the I_s . Second, let e be a white edge in T . It must be that e was an

edge between two K_3 s. We consider $T \vdash e$. We must use e as a black edge, but this gives us a P_3 using e and one black edge from an adjacent K_3 . Further, there are enough other disjoint K_3 s to take one vertex of each and form the I_3 . \square

This chapter has provided us with some nice bounds on the induced saturation number. We have also established that the induced saturation number can vary from zero up to the saturation number. What we have not accomplished in this chapter is to show that the induced saturation number can actually fall somewhere between zero and the saturation number. Our result on general paths leads us to believe that the induced saturation number of paths will fall somewhere in the middle of that range, but we have not proven that thus far. Thankfully, in the next chapter we will provide a result on the induced saturation number of paths on four vertices that will show the induced saturation number can be non-zero and strictly less than the saturation number.

CHAPTER 3. INDUCED SATURATION NUMBER OF P_4 IS $\lceil \frac{n+1}{3} \rceil$

This chapter is entirely devoted to proving that $\text{indsat}(n, P_4) = \lceil \frac{n+1}{3} \rceil$. This is a very exciting result, as in the previous chapter we saw that the induced saturation number and the saturation number could be different, but in that chapter the only graphs for which the induced saturation number was known it either matched the saturation number or was zero. This will be our only known induced saturation number that is not zero and which is strictly less than the saturation number. Proving that we have indeed found the appropriate induced saturation number for the path of length four will take up the majority of the next two chapters. The current chapter provides the bulk of the ‘meat’ of the proof. The subsequent chapter will provide a collection of facts and lemmas most of which are necessary, but few of which are overly exciting.

In the first section we review the important notation and definitions that we will use throughout. In the second section, we provide an upper bound, via construction, on the induced saturation number for P_4 . In the third section, we provide a lower bound, via case analysis and induction, on the induced saturation number for P_4 . In the final section, we provide a technical lemma which handles a bulk of the work needed to prove this result.

3.1 Definitions and Notation

This brief section is devoted to reviewing a few of the important definitions that we will use throughout the remaining of this chapter.

We start by recalling that we will denote a path on four vertices as P_4 and a complete graph on three vertices as K_3 or a triangle. A trigraph is a quadruple $(V(T); EB(T), EW(T), EG(T))$ in which $(EB(T), EW(T), EG(T))$ is a partition of the edges of the complete graph on the vertex set $V(T)$. (Possibly one or two of $EB(T), EW(T), EG(T)$ can be the empty set.) We

call the set $EB(T)$ the *black edges* of T , the set $EW(T)$ the *white edges* of T , and the set $EG(T)$ the *gray edges* of T . We will think of these as edges, non-edges, and ‘free’ edges, where ‘free’ means that we may regard the gray edges as either black or white. We note here that if $EG(T) = \emptyset$, then our trigraph may be regarded as just a graph, in which the black edges are treated as ‘edges’ and the white edges are treated as ‘nonedges’. We say a *realization* of a trigraph T is a graph G with $V(G) = V(T)$ and $E(G) = EB(T) \cup S$ for some subset S of $EG(T)$. That is, we may see this as setting some gray edges to be black and setting the remaining gray edges to white. We say the complement, \bar{T} of a trigraph T is a trigraph with $V(\bar{T}) = V(T)$, $EB(\bar{T}) = EW(T)$, $EG(\bar{T}) = EG(T)$, and $EW(\bar{T}) = EB(T)$, which we may view as extending the definition of graph complement.

We will be relaxed in our notation and say that a trigraph T has an induced copy of the graph H if there exists a realization of T such that an induced copy of H appears in that realization. We refer to the process of changing a black or white edge to gray as *flipping* the edge. We will use the notation $T \vdash e$ to mean that we are flipping the non-gray edge e to gray in the trigraph T .

We say a trigraph T is H -induced-saturated if a graph H does not appear in T as an induced subgraph, but when we flip a white or black edge of T to gray H appears as an induced subgraph. Further, we defined

$$\text{indsat}(n, H) = \min\{|EG(T)| : |V(T)| = n \text{ and } T \text{ is } H\text{-induced-saturated}\}$$

We use the short hand notation xy to mean the edge $\{x, y\}$. Further, we will use $xy \in EB(T)$ to mean that the edge xy is colored black in T . Likewise $xy \in EW(T)$ and $xy \in EG(T)$ for xy a white and gray edge respectively.

We extend standard graph notation to trigraphs. If V is a subset of vertices of a trigraph T , we will let $T[V]$ be the trigraph induced in T by the vertices V .

3.2 Upper Bound by Construction

This section is devoted to finding a trigraph that is P_4 -induced-saturated. In particular, we would like to find the graph with the fewest number of gray edges that is P_4 -induced-saturated. Consider the trigraphs T in Figures 3.1, 3.2, 3.3, where the trigraph T will depend on the modularity of n . For all the constructions, a cluster of bold edges will mean that the vertex to which they are connected is connected to every vertex to the right. As an example, in Figure 3.1, c_1 is connected to all a_j and b_j for $1 < j \leq k+1$, as well as all c_m for $1 < m \leq k$.

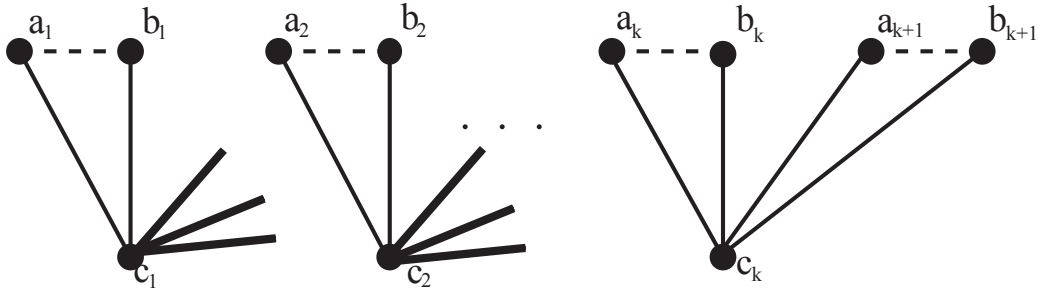


Figure 3.1: H when $n \equiv 2 \pmod{3}$

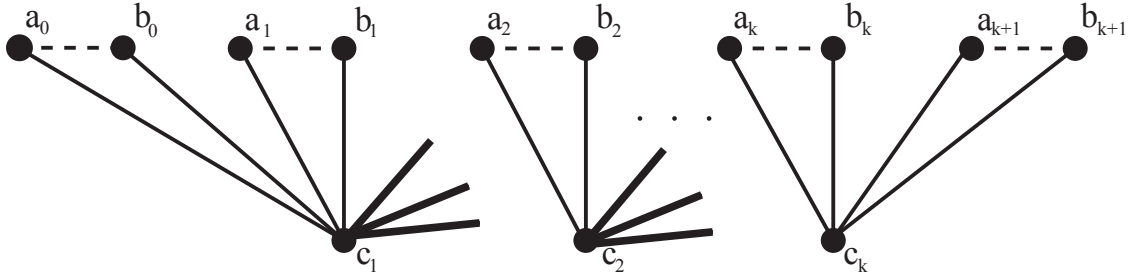


Figure 3.2: H when $n \equiv 1 \pmod{3}$

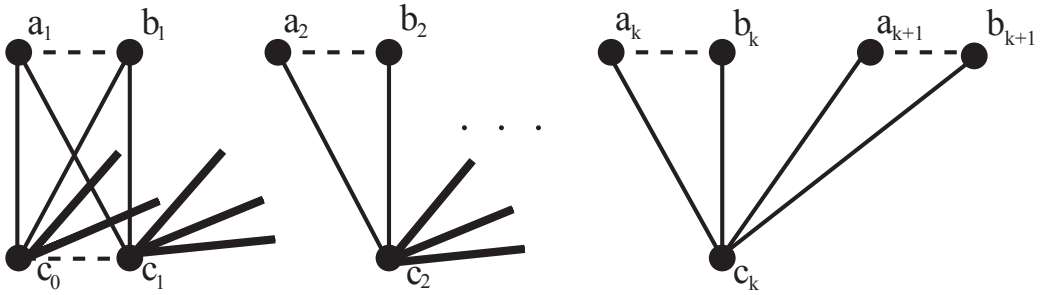


Figure 3.3: H when $n \equiv 0 \pmod{3}$

Theorem 3.2.1. $indsat(n, P_4) \leq \lceil \frac{n+1}{3} \rceil$.

Proof. To prove the above theorem we will show that the trigraph T as shown in Figures 3.1, 3.2, and 3.3 is P_4 -induced-saturated. We will base the majority of the proof off of the construction given in Figure 3.1, as it is a subgraph of the other two constructions. Whenever possible we will take advantage of the symmetry between a_i and b_i , by generally only considering a_i .

The first major step is to show that G does not contain an induced P_4 . To see this, we note that there can not be an induced P_4 of the form, a_i, c_i, x, y , where x and y are some vertices to the right of c_i , since c_i is connected to both x and y . Thus, if there were an induced P_4 it would have to be of the form a_i, b_i, c_i, y , but this can not happen as a_i and b_i are connected to c_i . In Figure 3.2, we also need to consider the potential P_4 , $a_0 b_0 c_1 y$, but this is not an induced P_4 as the edge $a_0 c_1$ is black. In Figure 3.3, we need to consider the potential P_4 , $c_0 a_1 c_1 y$, but this is not an induced P_4 , since the edge $c_0 y$ is black.

The second major step is to show that if any edge is flipped to gray then G does contain an induced P_4 . We will take care of this with case analysis, again focusing on Figure 3.1 first and then filling in the missing edges for the other two constructions.

Case 1 : {Flip $a_j c_i$ } We flip the edge $a_j c_i$ for $1 \leq i < j \leq k+1$ from black to gray. In this case, $a_i c_i b_j a_j$ is an induced P_4 in $T \div a_j c_i$.

Case 2 : {Flip $a_i c_i$ } We flip the edge $a_i c_i$ for $1 \leq i \leq k$ from black to gray. In this case, $a_i b_i c_i b_{i+1}$ is an induced P_4 in $T \div a_i c_i$.

Case 3 : {Flip $c_i c_j$ } We flip the edge $c_i c_j$ for $1 \leq i < j \leq k$ from black to gray. In this case, $a_i c_i a_j c_j$ is an induced P_4 in $T \div c_i c_j$.

Case 4 : {Flip $a_i a_j$ } We flip the edge $a_i a_j$ for $1 \leq i < j \leq k+1$ from white to gray. In this case, $b_i a_i a_j b_j$ is an induced P_4 in $T \div a_i a_j$.

Case 5 : {Flip $a_i c_j$ } We flip the edge $a_i c_j$ for $1 \leq i < j \leq k$ from white to gray. In this case, $b_i a_i c_j a_{j+1}$ is an induced P_4 in $T \div a_i c_j$.

The only additional case in Figure 3.2 is if the edge $a_0 c_1$ is flipped from black to gray, similar to case 2 above, $a_0 b_0 c_1 b_1$ is an induced P_4 in $T \div a_0 c_1$. We have two additional cases in Figure 3.3. The first is we flip the edge $a_1 c_0$ to gray. In this case, $a_1 c_1 b_1 c_0$ is an induced P_4 in $T \div a_1 c_0$. The second is we flip the edge $c_0 x$ where x is either a_i or c_j for $1 \leq i \leq k+1$ and $2 \leq j \leq k$ to gray. In this case, $c_0 b_1 c_1 x$ is an induced P_4 in $T \div c_0 x$.

Therefore we have shown that the trigraph T does not contain an induced P_4 , but whenever a black or white edge of T is flipped to gray, then an induced P_4 appears. This completes the proof of Theorem 3.2.1. \square

Therefore we have shown that there exist trigraphs for which the number of gray edges is $\lceil \frac{n+1}{3} \rceil$. All that remains to show is that it is impossible to find a trigraph which is P_4 -induced-saturated and has fewer than $\lceil \frac{n+1}{3} \rceil$ gray edges. In the next section with the help of the final section, we will show that this is the case.

3.3 Lower Bound by Induction

In this section we will present the core of the proof that we can not do any better than the constructions given in the previous section. Additional parts of the proofs are provided in the next section and the next chapter.

Theorem 3.3.1. $indsat(n, P_4) \geq \lceil \frac{n+1}{3} \rceil$.

Proof. We will prove this statement using strong induction. The base cases are proven in Lemmas 4.3.1, 4.3.2, 4.3.3. Therefore, our strong inductive hypothesis states, for a fixed $n \in \mathbb{N}$, if T is a P_4 -induced-saturated trigraph with $|V(T)| = m < n$ then $|EG(T)| \geq \lceil \frac{m+1}{3} \rceil$.

We note that by Fact 4.2.15, we are guaranteed that there exists a gray edge in T . Further, using Fact 4.2.2, the gray components of T are either K_3 's or S_k 's.

By Fact 4.2.6 each gray star, S , on at least two vertices, partitions $V(T)$ into sets S , X , Y and Z , any of which could be empty. Likewise, Fact 4.2.7 each gray triangle, R , partitions $V(T)$ into sets R , X , and Y , with $T[R, X]$ all black and $T[R, Y]$ all white either of which could be empty. In the case of a gray triangle we will assume a set Z exists but is the empty set.

Therefore, we can consolidate our cases and just consider what happens when we have a gray component, say C , which then defines sets X , Y , and Z . Now, we want to understand a little more about the set Z . The following claim establishes the behavior of Z with the sets X and Y .

Claim 3.3.2. *With X, Y, Z defined as above, the edges in $T[Z, X]$ are black and the edges in $T[Z, Y]$ are white.*

Proof. If C is a gray triangle or Z is otherwise empty, the claim is vacuous, so we assume that it is a star and Z is not empty. Without loss of generality using Fact 4.2.6, let us assume uv is a gray edge in the star such that the edges in $T[Z, \{u\}]$ are black and the edges in $T[Z, \{v\}]$ are white. Let $z \in Z$. If $x \in X$ and zx is white, then $zuxv$ is an induced P_4 in T as in Figure 3.4a, a contradiction. If $y \in Y$ and zy is black, then $yzuv$ is an induced P_4 in T as in Figure 3.4b, also a contradiction. This proves Claim 3.3.2. \square

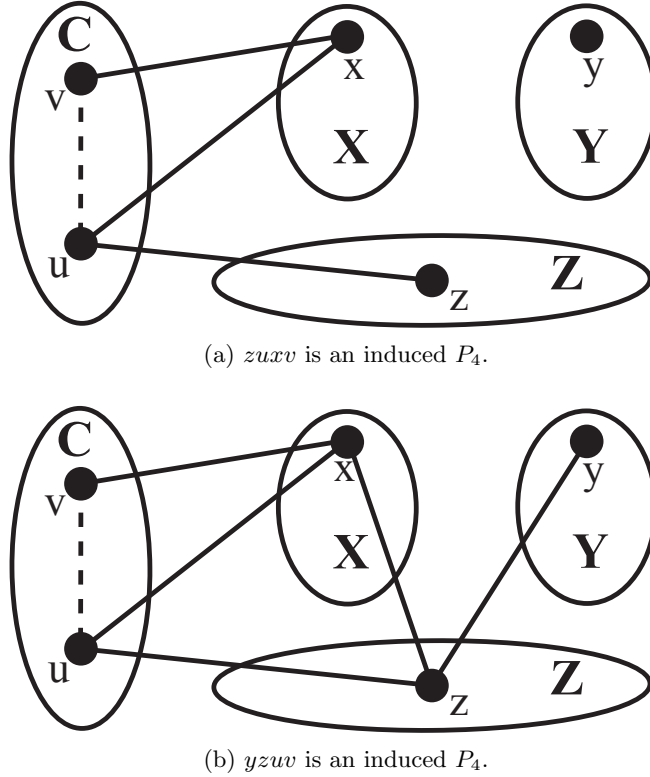


Figure 3.4: Two examples for the proof of Claim 3.3.2

At this point we are set up fairly nicely, but there is a chance that there are gray edges between the sets X and Y . Having gray edges between these sets isn't impossible to overcome, but the remainder of the proof is much cleaner if we can eliminate those gray cross edges. To this end, we let C_0 be a gray component in which Z is maximum-sized. Now we show that we have accomplished our goal.

Claim 3.3.3. *With X, Y, Z defined by C_0 , there are no gray edges in $T[X, Y]$.*

Proof. For sake of contradiction, let xy be a gray edge such that $x \in X$ and $y \in Y$. Further, let uv be a gray edge in C_0 . By Claim 3.3.2, the set $Z' = Z \cup \{u, v\}$ has the property that the edges in $T[\{x\}, Z']$ are black and the edges in $T[\{y\}, Z']$ are white. Hence, we have that Z' is strictly larger than the set Z that was formed by C_0 , as in Figure 3.5. This is our sought contradiction. \square

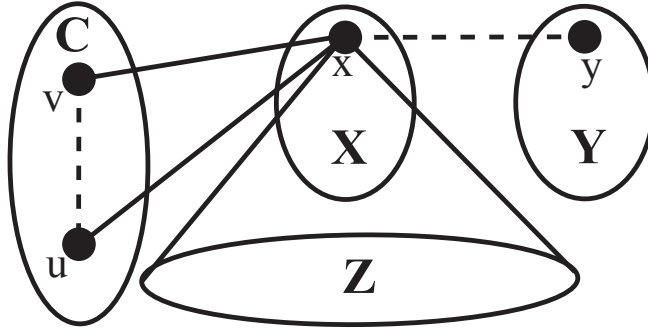


Figure 3.5: Figure for proof of Claim 3.3.3. A trigraph showing that the Z -set formed by xy is larger than the Z -set formed by $uv \in C_0$.

Now we are set up to apply Lemma 3.4.1 to finish the proof. To see this note the following: Condition (1) holds by Claim 3.3.3. Condition (2) holds because of the definition of X and Y from Facts 4.2.6 and 4.2.7 as well as because of Claim 3.3.3. Condition (3) holds because of the existence of C_0 .

Let uv be a gray edge in C_0 such that the edges in $T[Z, \{u\}]$ are black and the edges in $T[Z, \{v\}]$ are white. We have several cases to consider.

Case (a.) $|Z| = 1$:

We will show that this case cannot occur. Suppose it does and let $Z = \{z\}$. Consider a realization of P_4 , call it P , in the trigraph $T \div zv$. By the definition of P , it must use zv as a black edge. We will show first that P cannot contain an $x \in X$. If it did, then we know $xz \in EB(T)$ and $xv \in EB(T)$. This means xvz creates a black triangle, so P can not be an induced P_4 . Second, we will show that P cannot contain a member of Y either. This is clear because there is no black/gray path from a vertex in Y to either z or v that avoids X .

Since P cannot contain a vertex in $X \cup Y$, it must be contained in the vertices $\{z, u, v\}$, but there are only three of those, a contradiction. So, Case (a.) cannot occur.

Case (b.) $Z = \emptyset$, $|X| + |Y| \leq 1$:

In this case, T is almost entirely the gray component C_0 . We have $|V(T)| = |C_0| + 1$. We assume T has at least 4 vertices, so C_0 is not a trivial gray component. Therefore, we know C_0 is either a gray star or a gray triangle and hence it has at least $|C_0| - 1 = n - 2$ gray edges. Since $n - 2 \geq \lceil (n + 1)/3 \rceil$ for all $n \geq 4$, Case (b.) satisfies the conditions of the theorem.

Case (c.) $Z = \emptyset$, $|X| + |Y| \geq 2$:

In this case, since $|X| + |Y| \geq 2$, we can apply Lemma 3.4.1 to get that there are at least $\left\lceil \frac{|X| + |Y|}{3} \right\rceil$ gray edges in $X \cup Y$. Further, as in case (b.), we know there are at least $|C_0| - 1$ gray edges in C_0 . Therefore the total number of gray edges in T is at least

$$\left\lceil \frac{|X| + |Y|}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{n + 2|C_0| - 3}{3} \right\rceil \geq \left\lceil \frac{n + 1}{3} \right\rceil,$$

where the first equality holds since $|V(T)| = |X| + |Y| + |C_0|$ and the inequality holds since we assume C_0 is not trivial, that is $|C_0| \geq 2$. Thus, Case (c.) satisfies the conditions of the theorem.

Case (d.) $|Z| \geq 2$, $|X| + |Y| \leq 1$:

Since the set Z only occurs when C_0 is a gray star, we know by Fact 4.2.6 that each $z \in Z$ has the same black neighborhood and white neighborhood in C_0 . Further, from Claim 3.3.2, we know that each $z \in Z$ has all black edges to X and all white edges to Y . Therefore by Fact 4.2.13 (where $S = Z$) the subtrigraph $T[Z]$ is P_4 -induced-saturated. By the inductive hypothesis, the number of gray edges in $T[Z]$ is at least $\left\lceil \frac{|Z| + 1}{3} \right\rceil$. Also, as before the number of gray edges in C_0 is at least $|C_0| - 1$. Hence, the number of gray edges in T is at least

$$\left\lceil \frac{|Z| + 1}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{|Z| + 3|C_0| - 2}{3} \right\rceil = \left\lceil \frac{n + 2|C_0| - 3}{3} \right\rceil \geq \left\lceil \frac{n + 1}{3} \right\rceil,$$

where the second equality holds since $|V(T)| \leq |C_0| + |Z| + 1$ and the inequality holds since we assume C_0 is not trivial, that is $|C_0| \geq 2$. Thus, Case (d.) satisfies the conditions of the theorem.

Case (e.) $|Z| \geq 2$, $|X| + |Y| \geq 2$:

Again, since the set Z only occurs when C_0 is a gray star, we know by Fact 4.2.6 that each $z \in Z$ has the same black neighborhood and white neighborhood in C_0 . Further, from Claim 3.3.2, we know that each $z \in Z$ has all black edges to X and all white edges to Y . Therefore by Fact 4.2.13 (where $S = Z$) the subtrigraph $T[Z]$ is P_4 -induced-saturated. By the inductive hypothesis, the number of gray edges in $T[Z]$ is at least $\left\lceil \frac{|Z|+1}{3} \right\rceil$. Also again, since $|X| + |Y| \geq 2$, we can apply Lemma 3.4.1 to get that there are at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ gray edges in $X \cup Y$. Therefore the total number of gray edges in T is at least

$$\left\lceil \frac{|Z|+1}{3} \right\rceil + \left\lceil \frac{|X|+|Y|}{3} \right\rceil + |C_0| - 1 = \left\lceil \frac{|X|+|Y|+|Z|+1+3|C_0|-3}{3} \right\rceil = \left\lceil \frac{n+2|C_0|-2}{3} \right\rceil \geq \left\lceil \frac{n+2}{3} \right\rceil,$$

where the second equality holds since $|V(T)| = |X| + |Y| + |Z| + |C_0|$ and the inequality holds since we assume C_0 is not trivial, that is $|C_0| \geq 2$. Thus, Case (e.) satisfies the conditions of the theorem. So, the proof of Theorem 3.3.1 is complete. □

As stated this is just the core of the proof, using induction and case analysis. In the next section we state and prove the technical lemma.

3.4 Technical Lemma

Lemma 3.4.1. *Let T be a trigraph which is P_4 -induced-saturated with $|V(T)| = n$ such that $V(T) = W \cup X \cup Y$ and $V(T) \neq W$ such that the following is true:*

1. $T[X, Y]$ has no gray edge.
2. Each edge in $T[X, W]$ is black and each edge in $T[Y, W]$ is white.
3. There exist vertices $u, v \in W$ such that $uv \in EG(T)$.

If $|X| + |Y| \geq 2$, then the number of gray edges in $T[X \cup Y]$ is at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$.

Proof. To start the proof, we make two notes. First, we are still operating under the inductive hypothesis as stated in the proof of Theorem 3.3.1. Second, neither X nor Y is empty. If so,

say $X = \emptyset$, then $|Y| \geq 2$ and the inductive hypothesis gives us that there are $\lceil \frac{n+1}{3} \rceil$ gray edges in $X \cup Y$.

Throughout this proof, we will be using our non-standard definition of a component. We will call the set of vertices which are connected via black or gray paths a black/gray component. It does not imply that there are no white edges in the component. Likewise the set of vertices which are connected via white or gray paths will be called a white/gray component.

We would like to understand the structure between X and Y . We hope to show that their neighborhoods nest. We partition the vertices of X into equivalence classes according to their neighborhoods in Y . For each $x \in X$, denote the set of $y \in Y$ such that the edge xy is black as $N_Y^B(x)$.

The next claim and corollary establishes that every vertex in a white and gray component of X behaves the same to black and gray components in Y .

Claim 3.4.2. *If $x_1x_2 \in EW(T) \cup EG(T)$ and $x_1y \in EB(T) \cup EG(T)$, then $x_2y \in EB(T)$ which implies $x_1y \in EB(T)$.*

Proof. Let $x_1x_2 \in EW(T) \cup EG(T)$ and $x_1y \in EB(T) \cup EG(T)$. In this case, if $x_2y \in EW(T)$, then yx_1ux_2 is an induced P_4 in T as in Figure 3.6. Therefore, it must be that $x_2y \in EB(T)$. Now, if $x_2y \in EB(T)$ and $x_1y \in EG(T)$, then yx_2ux_1 is an induced P_4 in T . Therefore, it must be that $x_1y \in EB(T)$. \square

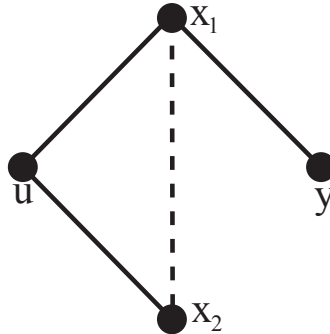


Figure 3.6: Figure for proof of Claim 3.4.2. Assuming $x_1x_2 \in EW(T) \cup EG(T)$, yx_2ux_1 is an induced P_4 in T .

Corollary 3.4.3. *If $x, x' \in X$ have different neighborhoods in Y , then xx' is black.*

Proof. This follows directly by multiple applications of Claim 3.4.2. \square

The next claim and corollary establishes that every vertex in a black and gray component of Y behaves the same to white and gray components in X .

Claim 3.4.4. *If $y_1y_2 \in EG(T) \cup EB(T)$ and $xy_1 \in EB(T) \cup EG(T)$, then $xy_2 \in EB(T)$ which implies $xy_1 \in EB(T)$.*

Proof. Let $y_1y_2 \in EG(T) \cup EB(T)$ and $xy_1 \in EB(T) \cup EG(T)$. In this case, if $xy_2 \in EW(T)$, y_2y_1xu is an induced P_4 in T as in Figure 3.7. Therefore, it must be that $xy_2 \in EB(T)$. Now, if $xy_2 \in EB(T)$ and $xy_1 \in EG(T)$, then y_1y_2xu is an induced P_4 in T . Therefore, it must be that $xy_1 \in EB(T)$. \square

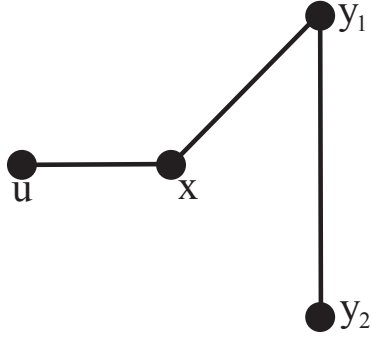


Figure 3.7: Figure for proof of Claim 3.4.4. Assuming $y_1y_2 \in EG(T) \cup EB(T)$, y_2y_1xu is an induced P_4 in T .

Corollary 3.4.5. *If $y, y' \in Y$ have different neighborhoods in X , then yy' is white.*

Proof. This follows directly by multiple applications of Claim 3.4.4. \square

Now that we know the edges between equivalence classes of X are black and between equivalence classes of Y are white, we would like to show that the neighborhoods of these white/gray and black/gray components nest. Let X_1, X_2, \dots, X_m be the white/gray components in X and Y_1, Y_2, \dots, Y_n be the black/gray components in Y .

Claim 3.4.6. $N_X^b(Y_i)$ are nested.

Proof. For sake of contradiction we'll assume they are not nested. Let $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$, $y_2 \in Y_2$, and $x_1y_1 \in EB(T)$, $x_2y_2 \in EB(T)$, but $x_1y_2 \in EW(T)$ and $x_2y_1 \in EW(T)$. That is x_1 is a neighbor of Y_1 but not Y_2 and x_2 is a neighbor of Y_2 but not Y_1 . Now as X_1 and X_2 are different white/gray components it must be that $x_1x_2 \in EB(T)$. Likewise as Y_1 and Y_2 are different black/gray components it must be that $y_1y_2 \in EW(T)$. Therefore, we have the desired contradiction as $y_1x_1x_2y_2$ is an induced P_4 in T . as in Figure 3.8. \square

Claim 3.4.7. $N_Y^b(X_i)$ are nested.

Proof. For sake of contradiction we'll assume they are not nested. Let $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$, $y_2 \in Y_2$, and $x_1y_1 \in EB(T)$, $x_2y_2 \in EB(T)$, but $x_1y_2 \in EW(T)$ and $x_2y_1 \in EW(T)$. That is y_1 is a neighbor of X_1 but not X_2 and y_2 is a neighbor of X_2 but not X_1 . Now as X_1 and X_2 are different white/gray components it must be that $x_1x_2 \in EB(T)$. Likewise as Y_1 and Y_2 are different black/gray components it must be that $y_1y_2 \in EW(T)$. Therefore, we have the desired contradiction as $y_1x_1x_2y_2$ is an induced P_4 in T as in Figure 3.8. \square

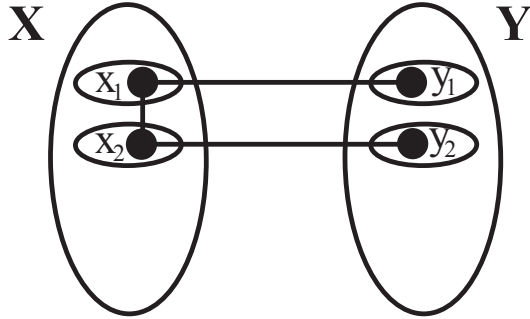


Figure 3.8: Figure for proof of Claim 3.4.6 and 3.4.7. Assuming neighborhoods are not nested, $y_1x_1x_2y_2$ is an induced P_4 in T .

As we now know that the neighborhoods are nested, we are going to slightly change our notation. We once again define an equivalence relation on the vertices in X so that $x, x' \in X$ are equivalent if and only if $N_Y^B(x) = N_Y^B(x')$. Let the equivalence classes be X_1, X_2, \dots, X_ℓ . These equivalence classes of X , form equivalence classes in Y . For $j = 1, \dots, \ell + 1$, the set Y_j has the property that the edges in $T[X_i, Y_j]$ are white for $i = 1, \dots, j - 1$ and the edges in $T[X_i, Y_j]$

are black for $i = j, \dots, \ell$, as in Figure 3.9. Further, we know that the X_i are a collection of white/gray components in X and the Y_i are a collection of black/gray components in Y .

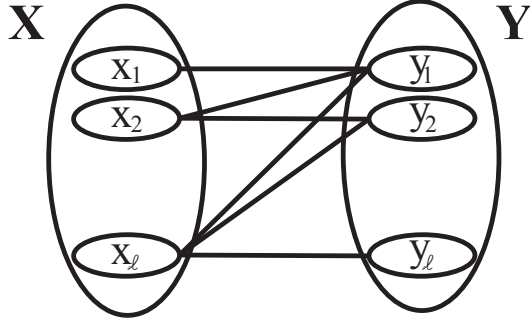


Figure 3.9: A decomposition of X and Y into equivalence classes. The pair (X_i, Y_j) consists of black edges if $i \geq j$; otherwise, it consists of white edges.

Note here that by definition each of the Y_i for $2 \leq i \leq \ell$ are not empty. If Y_j is empty, then $N_Y^B(x) = Y_1 \cup \dots \cup Y_{j-1}$ for all $x \in X_{j-1} \cup X_j$, which is a contradiction to the way we chose our equivalence classes. From this point on our goal is to consider X_i and Y_i together and count the gray edges in their union. Once again we provide a series of small claims.

Claim 3.4.8. *For $i = 1, \dots, \ell$, both X_i and Y_i are P_4 -induced-saturated.*

Proof. Since every vertex in $\overline{X_i}$ is either adjacent to every vertex in X_i in black or is adjacent to every vertex in X_i in white, Fact 4.2.13 with $X_i = S$, gives that X_i must be P_4 -induced-saturated. Likewise for each Y_j . \square

To be able to consider X_i and Y_i together we need to know something about their relative sizes. We make a statement to this regard.

Claim 3.4.9. *For $i = 1, \dots, \ell$, if $|X_i| = 1$, then $|Y_i| \neq 1$.*

Proof. Assume for sake of contradiction that $\{y_i\} = Y_i$. Let $\{x_i\} = X_i$ and $e = x_i y_i$.

Let P be an induced P_4 in $T \div e$. Since e was a black edge, it must be used as white in P . In P , y_i must have a black neighbor. By definition of Y_i , the only possible black neighbor of y_i is an $x \in X_{i+1} \cup \dots \cup X_\ell$. We know $x x_i \in EB(T)$. Let z be a vertex that is a black neighbor of x_i , that is $x_i z \in EB(T)$. We have that $x z \in EB(T)$ from Claim 3.4.7. Therefore, for any such

z, xzx_i induces a black triangle. Thus, there is no way for P to be an induced P_4 , which is a contradiction. Therefore, as desired it must be that $|Y_i| \geq 2$. \square

We make two notes about the above claim. First, the claim does not guarantee that $Y_i \neq \emptyset$, so there is still a chance that Y_1 and $Y_{\ell+1}$ are the empty set. Second, the contrapositive shows that if $|Y_i| = 1$, then $|X_i| \neq 1$. To help in the case when $Y_1 = \emptyset$, we make a statement about the relative sizes of X_i and Y_{i+1} .

Claim 3.4.10. *For $i = 1, \dots, \ell$, if $|X_i| = 1$, then $|Y_{i+1}| \neq 1$.*

Proof. Assume for sake of contradiction that $\{y_{i+1}\} = Y_{i+1}$. Let $\{x_i\} = X_i$ and $e = x_i y_{i+1}$. Let P be an induced P_4 in $T \div e$. Since e was a white edge, it must be used as black in P . It must be the case that y_{i+1} is a leaf of P , since any black neighbor of Y_{i+1} is a black neighbor of x_i . Thus, P must be comparable to the form $z_1 z_2 x_i y_{i+1}$, but by inspection it is clear that if $z_2 x_i \in EB(T)$, then $z_1 x_i \in EB(T)$, which means P is not an induced P_4 . This is the sought contradiction. \square

We are now close to completing the proof of the technical lemma, but as we show in the following claim we will still need to consider two special cases.

Claim 3.4.11. *There are at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ gray edges unless one of the following cases occurs:*

- (i.) $|Y_1| = \dots = |Y_{\ell+1}| = 1$ and $|X_1|, \dots, |X_\ell| \geq 2$.
- (ii.) $|X_1| = \dots = |X_\ell| = 1$ and $|Y_2|, \dots, |Y_\ell| \geq 2$, and $Y_1 = Y_{\ell+1} = \emptyset$.

Proof. First, suppose $Y_1 \neq \emptyset$. Consider the pairs (X_i, Y_i) for $i = 1, \dots, \ell$. By Claim 3.4.9, at least one of the sets must have size at least 2. If both have size at least 2, then the number of gray edges in $X_i \cup Y_i$ is at least $\left\lceil \frac{|X_i|+1}{3} \right\rceil + \left\lceil \frac{|Y_i|+1}{3} \right\rceil \geq \left\lceil \frac{|X_i|+|Y_i|+1+1}{3} \right\rceil$. If not, say $|Y_i| \geq 2$, then the number of gray edges in $X_i \cup Y_i$ is at least $\left\lceil \frac{|Y_i|+1}{3} \right\rceil = \left\lceil \frac{|X_i|+|Y_i|}{3} \right\rceil$. So, in this case, the total number of gray edges in $X \cup Y$ is at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ unless $|Y_{\ell+1}| = 1$ and there are ℓ other components of size 1. By Claims 3.4.9 and 3.4.10, this can only occur if $|Y_1| = \dots = |Y_\ell| = 1$. This is precisely case (i.).

Second, suppose $Y_1 = \emptyset$. Consider the pairs (X_i, Y_{i+1}) for $i = 1, \dots, \ell$. By Claim 3.4.10, at least one of the sets must have size at least 2. If both have size at least 2, then the number of gray edges in $X_i \cup Y_{i+1}$ is at least $\left\lceil \frac{|X_i|+1}{3} \right\rceil + \left\lceil \frac{|Y_{i+1}|+1}{3} \right\rceil \geq \left\lceil \frac{|X_i|+|Y_{i+1}|+1+1}{3} \right\rceil$. If not, say $|Y_{i+1}| \geq 2$, then the number of gray edges in $X_i \cup Y_{i+1}$ is at least $\left\lceil \frac{|Y_{i+1}|+1}{3} \right\rceil = \left\lceil \frac{|X_i|+|Y_{i+1}|}{3} \right\rceil$. So, in this case, the total number of gray edges in $X \cup Y$ is at least $\left\lceil \frac{|X|+|Y|}{3} \right\rceil$ unless $|X_1| = 1$ and there are $\ell - 1$ other components of size 1. By Claims 3.4.9 and 3.4.10, this can only occur if $|X_2| = \dots = |X_\ell| = 1$. This is precisely case (ii). \square

Now we are very close to finished with this proof. We only have two cases left to consider. As they are challenging, we will look at each case individually.

Case (i.) As a guide, throughout this discussion refer to Figure 3.10. Consider the trigraph, T' , induced by $V(T) - (X_\ell \cup Y_{\ell+1})$. We claim that T' is P_4 -induced-saturated. Suppose not and consider $T \div e$ such that e is a white or black edge with both endpoints in $V(T')$.

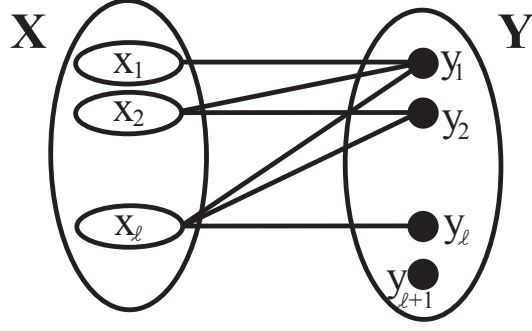
First, we show that no realization of P_4 in T can have three of its vertices in $V(T')$. We see this because the pair $(Y_{\ell+1}, V(T'))$ has only white edges, giving a vertex of degree 0, and $(X_\ell, V(T'))$ has only black edges, giving a vertex of degree 3. As an induced P_4 doesn't contain a vertex of degree 0 or of degree 3, we are done with this case. Second, we show that no realization of P_4 in T can have two vertices in $V(T')$ and two in X_ℓ . This is true since each $x_\ell \in X_\ell$ is connected to $v \in V(T')$ by black edges. If we used two vertices from X_ℓ we would induce a C_4 , which forbids a P_4 . Finally, we show that the realization cannot have $x_\ell \in X_\ell$ and the vertex in $Y_{\ell+1}$. If it did, x_ℓ would have degree 3 once again a contradiction. Therefore, the realization must have all four vertices in $V(T')$, which shows that T' is P_4 -induced-saturated.

Hence, $V(T')$ is a P_4 -induced-saturated trigraph on at least 2 vertices and by the inductive hypothesis, the number of gray edges in T is at least

$$\left\lceil \frac{|V(T')|+1}{3} \right\rceil + \left\lceil \frac{|X_\ell|+1}{3} \right\rceil \geq \left\lceil \frac{|V(T')|+|X_\ell|+2}{3} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil,$$

where the equality holds since $|V(T)| = |V(T')| + |X_\ell| + |Y_{\ell+1}| = |V(T')| + |X_\ell| + 1$.

Case (ii.) For reference throughout this discussion refer to Figure 3.11. Consider the trigraph, T'' , induced by $V(T) - (X_\ell \cup Y_\ell)$. We claim that T'' is P_4 -induced-saturated. Suppose not and consider $T \div e$ such that e is a white or black edge with both endpoints in $V(T'')$. First, we

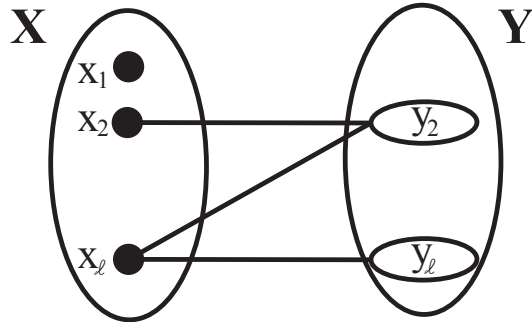
Figure 3.10: The trigraph with each Y_i being of size 1.

show that no realization of P_4 in T can have three of its vertices in $V(T'')$. We see this because the pair $(X_\ell, V(T''))$ has only black edges, giving a vertex of degree 3, and $(Y_\ell, V(T''))$ has only white edges, giving a vertex of degree 0. As an induced P_4 doesn't contain a vertex of degree 3 or of degree 0, we are done with this case. Second, we show that no realization of P_4 in T can have two vertices in $V(T'')$ and two in Y_ℓ . This is true since each $y_\ell \in Y_\ell$ is connected to $v \in V(T'')$ by white edges. If we used two vertices from Y_ℓ we would induce a $\overline{C_4}$, which forbids a P_4 . Finally, we show the realization cannot have a vertex in Y_ℓ and the vertex x_ℓ in X_ℓ . If it did, x_ℓ would have degree 3 once again a contradiction. Therefore, the realization must have all four vertices in $V(T'')$, which shows that T'' is P_4 -induced-saturated.

Hence, $V(T'')$ a P_4 -induced-saturated trigraph on at least 2 vertices and by the inductive hypothesis, the number of gray edges in T is at least

$$\left\lceil \frac{|V(T'')| + 1}{3} \right\rceil + \left\lceil \frac{|Y_\ell| + 1}{3} \right\rceil \geq \left\lceil \frac{|V(T'')| + |Y_\ell| + 2}{3} \right\rceil = \left\lceil \frac{n+1}{3} \right\rceil,$$

where the equality holds since $|V(T)| = |V(T'')| + |X_\ell| + |Y_\ell| = |V(T'')| + |Y_\ell| + 1$.

Figure 3.11: The trigraph with each X_i being of size 1.

Having wrapped up these final two cases, the proof of Lemma 3.4.1 is complete.

□

In this chapter we have provided the core of the proof that $\text{indsat}(n, P_4) = \left\lceil \frac{n+1}{3} \right\rceil$, by providing an upper bound, a lower bound, and a technical lemma to support the proof of the lower bound. We still need to provide the proof of a few facts that were used in these proofs as well as the base cases that will allow our strong induction a place to start. All of these proofs will be provided in the next chapter.

CHAPTER 4. SUPPORTING MATERIAL FOR INDUCED SATURATION NUMBER OF P_4

This chapter is devoted to providing supporting materials for the proofs in Chapter 3. We will provide many facts, some of the facts were directly referenced in the previous chapter, some are referenced by other facts, some are referenced by the base cases, and a few are not referenced but are interesting in their own right. The base cases are proven using a brute force case analysis. As a result these proofs are long and tedious, which is part of the reason why they are separated from the proof.

The remainder of the chapter is organized as follows: in the first section we will review the required definitions and notation, the second section will be a collection of facts, and the final section will be proofs of the base cases used in Chapter 3.

4.1 Definitions and Notation

This brief section is devoted to reviewing a few of the important definitions that we will use throughout the remaining of this chapter.

We start by recalling that we will denote a path on four vertices as P_4 and a complete graph on three vertices as K_3 or a triangle. Recall the definitions in Section 1.1.1: A trigraph T is a quadruple $(V(T); EB(T), EW(T), EG(T))$ for which $(EB(T), EW(T), EG(T))$ is a partition of the edges of the complete graph on the vertex set $V(T)$. We call $EB(T)$, $EW(T)$ and $EG(T)$ the black, white and gray edge sets, respectively. A *realization* of a trigraph T is a graph G with $V(G) = V(T)$ and $E(G) = EB(T) \cup S$ for some subset S of $EG(T)$.

We will be relaxed in our notation and say that a trigraph T has an induced copy of the graph H if there exists a realization of T such that an induced copy of H appears in that

realization. We refer to the process of changing a black or white edge to gray as *flipping* the edge. We will use the notation $T \vdash e$ to mean that we are flipping the edge e to gray in the trigraph T .

Recall also from Section 1.1.1 that we said a trigraph T was H -induced saturated if a graph H does not appear in T as an induced subgraph, but when we flip an white or black edge of T to gray H appears as an induced subgraph. Further, we defined

$$\text{indsat}(n, H) = \min\{|EG(T)| : |V(T)| = n \text{ and } T \text{ is } H\text{-induced-saturated}\}.$$

We use the short hand notation xy to mean the edge $\{x, y\}$. Further, we will use $xy \in EB(T)$ to mean that the edge xy is colored black in T . Likewise $xy \in EW(T)$ and $xy \in EG(T)$ for xy a white and gray edge respectively.

We extend standard graph notation to trigraphs. If V is a subset of vertices of a trigraph T , we will let $T[V]$ be the trigraph induced in T by the vertices V . Also, the *complement*, \overline{T} of a trigraph T is a trigraph with $V(\overline{T}) = V(T)$, $EB(\overline{T}) = EW(T)$, $EG(\overline{T}) = EG(T)$, and $EW(\overline{T}) = EB(T)$.

We will call a vertex u in a trigraph T *universal* if every other vertex in T is adjacent to u via a black edge. Similarly, we call a vertex w in a trigraph T *isolated* if every other vertex in T is adjacent to w via a white edge.

4.2 Facts

In this section we will provide supporting facts and their proofs. Throughout this section, we will limit the number of cases by using the fact that P_4 is self-complementary, that is $\overline{P_4} = P_4$. This is the driving force behind our first fact.

Fact 4.2.1. *If a trigraph T is P_4 -induced-saturated, then the complement of T , \overline{T} , is P_4 -induced-saturated.*

Proof. Let T be a P_4 -induced-saturated trigraph. We note that if P , a P_4 , is an induced subgraph of a realization R of T , then \overline{P} , also a P_4 , is an induced subgraph of \overline{R} . Thus, it suffices to show that \overline{R} is a realization of \overline{T} . To this end, we see that the white edges in R were

either white or gray in T , so are black or gray in \overline{T} so can be chosen to be black. Likewise, the black edges in R were either black or gray in T , so are white or gray in \overline{T} so can be chosen to be white. Thus, \overline{R} is a realization of \overline{T} . \square

The next fact was crucial for us to be able to consider the gray components in a trigraph T which is P_4 -induced-saturated.

Fact 4.2.2. *If T is a P_4 -induced-saturated trigraph, then each gray component in T is either a K_3 or S_k on $k \geq 2$ vertices.*

Proof. This fact is equivalent to saying that there can not be a gray P_4 in a P_4 -induced-saturated trigraph. For sake of contradiction, we let T be a trigraph on four vertices which contains a gray P_4 , with vertices a, b, c, d . We will consider multiple cases, and show in each case how to find an induced P_4 in T . As T has only four vertices and contains a gray P_4 , we only have three non-gray edges to worry about. By Fact 4.2.1, we only need to consider the cases where 0 or 1 of the remaining 3 edges is black. Further, we use symmetry of the P_4 to reduce the cases with a single black edge to only two cases.

Case 1 : {Zero black edges}. As in Figure 4.1a, no non-gray edges are black. In this case $abcd$ is an induced P_4 in T , a contradiction.

Case 2 : {One black edge. Edge connecting endpoints is black}. As in Figure 4.1b, the edge connecting the end points of the path is black, that is $ad \in EB(T)$. In this case $dabc$ is an induced P_4 in T , a contradiction.

Case 3 : {One black edge. Edge connecting endpoint and non-endpoint is black}. As in Figure 4.1c, the edge connecting the first and third vertices of the path is black, that is $bd \in EB(T)$. In this case $abdc$ is an induced P_4 in T , a contradiction. \square

Fact 4.2.3. *If a trigraph T is P_4 -induced-saturated, then T does not have a black pendant edge.*

Proof. Let T be a trigraph which is P_4 -induced-saturated. Assume T has a pendant edge, that is $uv \in EB(G)$ and all other edges incident to u are white or gray. Let P be an induced P_4 which occurs in $T \cdot uv$. We know that P must use u and v . Further, we must use the edge uv as a white edge. Thus in $T \cdot uv$, u only has white and gray neighbors. Therefore if P contains

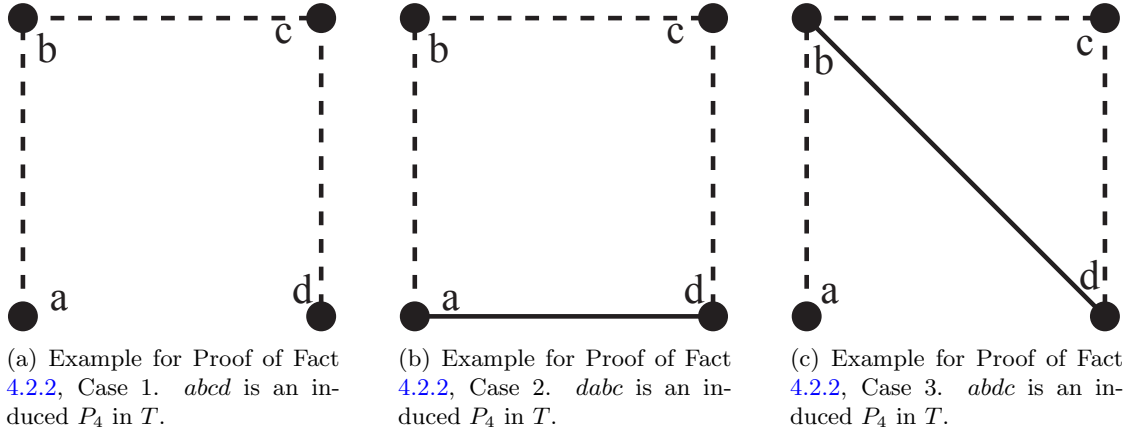


Figure 4.1: Examples of trigraphs which contain a gray P_4 .

u , then P must use gray edges that were in T originally. Hence, P was an induced P_4 in the original trigraph T , which is a contradiction. \square

The next two facts are very important. They are used to establish Fact 4.2.6, which shows how a gray star partitions the rest of the vertices in a P_4 -induced-saturated trigraph.

Fact 4.2.4. *Let T be a trigraph that is P_4 -induced-saturated. If $\{u, v_1, v_2, \dots, v_{k-1}\}$ is a gray star in T with center u , then every edge $\{v_i, v_j\}$ for $1 \leq i < j \leq k-1$ is the same.*

Proof. Let T be a trigraph which is P_4 -induced-saturated and contains S a gray star with center u and leaves v_1, v_2, v_3 . Assume for sake of contradiction that $v_1v_2, v_1v_3 \in EW(T)$ and $v_2v_3 \in EB(T)$. In this case, $v_1uv_3v_2$ is an induced P_4 in T , which is a contradiction to the fact that T is P_4 -induced-saturated. This is illustrated in Figure 4.2. We note here that by applying symmetry and complements the proof is complete. \square

Fact 4.2.5. *Let T be a trigraph that is P_4 -induced-saturated. If $\{u, v_1, v_2, \dots, v_{k-1}\}$ is a gray star in T with center u and $x \notin \{u, v_1, \dots, v_{k-1}\}$ then every edge $\{x, v_i\}$ for $1 \leq i \leq k-1$ is the same.*

Proof. Let T be a trigraph which is P_4 -induced-saturated and contains S a gray star with center u and leaves v_1, v_2 in addition to a vertex, x not in S . By Fact 4.2.4, we may assume $v_1v_2 \in EB(T)$ without loss of generality. We have two cases to consider. First, the edge

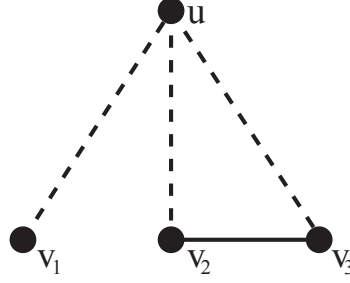


Figure 4.2: An example for the proof of Fact 4.2.4. A gray star with induced P_4 namely $v_1uv_3v_2$.

$ux \in EB(T)$, that is x is adjacent to the center of S . Assume for sake of contradiction that $v_1v_2, v_1x \in EB(T)$ and $v_2x \in EW(T)$. In this case, uxv_1v_2 is an induced P_4 in T , which is a contradiction to the fact that T is P_4 -induced-saturated. This is illustrated in Figure 4.3a. The second case to consider is when the edge $ux \in EW(T)$, that is x is not adjacent to the center of S . Assume for sake of contradiction that $v_1v_2, v_1x \in EB(T)$ and $v_2x \in EW(T)$. In this case, xv_1v_2u is an induced P_4 in T , which is a contradiction to the fact that T is P_4 -induced-saturated. This is illustrated in Figure 4.3b. We note here that by applying symmetry and complements the proof is complete. \square

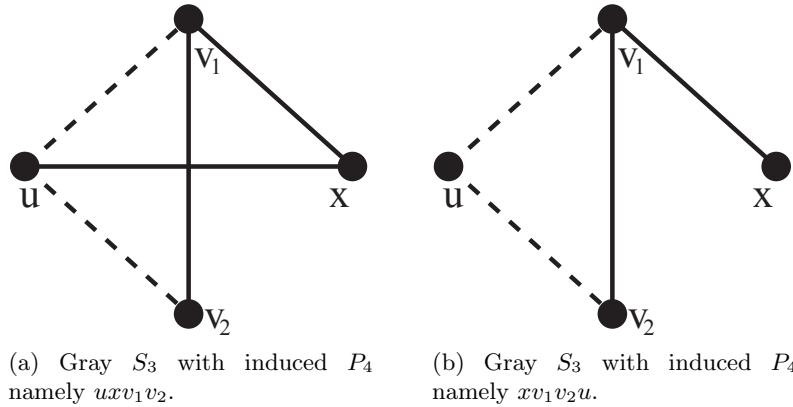


Figure 4.3: Two examples for the proof of Fact 4.2.5.

Fact 4.2.6. *Let T be a trigraph that is P_4 -induced-saturated. If $k \geq 2$ and $\{u, v_1, v_2, \dots, v_{k-1}\}$ is a gray star in T with center u , then $V(T) - \{u, v_1, v_2, \dots, v_{k-1}\}$ can be partitioned into sets X, Y and Z such that the following occur:*

- The edges xu and xv_i are black for all $i \in \{1, \dots, k-1\}$ and for all $x \in X$.
- The edges yu and yv_i are white for all $i \in \{1, \dots, k-1\}$ and for all $y \in Y$.
- One of the following occurs:
 - The edges $v_i v_j$ and $v_i z$ are white and uz are black for all distinct $i, j \in \{1, \dots, k-1\}$ and for all $z \in Z$.
 - The edges $v_i v_j$ and $v_i z$ are black and uz are white for all distinct $i, j \in \{1, \dots, k-1\}$ and for all $z \in Z$.

Proof. Let T be a trigraph which is P_4 -induced-saturated and contains a gray star $S = \{u, v_1, \dots, v_{k-1}\}$. From Fact 4.2.5, we know that for every vertex $w \in V(T) - S$, the edges wv_i have the same color. Now let $X = \{x \in V(T) - S : T[\{x\}, S] \text{ is black}\}$, let $Y = \{y \in V(T) - S : T[\{y\}, S] \text{ is white}\}$ and let $Z = V(T) - (X \cup Y \cup S)$. For each $z \in Z$, either both the edge zu is white and $T[\{z\}, S - \{u\}]$ is black or both the edge zu is black and $T[\{z\}, S - \{u\}]$ is white.

Now we must show that the vertices in Z behave as stated. If $k \geq 3$, we may assume without loss of generality that the edges of $T[\{v_1, \dots, v_{k-1}\}]$ are white. If $z \in Z$ and $uz \in EW(T)$ and $T[\{z\}, S - \{u\}]$ is black, then $v_1 z v_2 u$ is a realization of P_4 , a contradiction. Therefore it must be that all $z \in Z$ have the property that $T[\{z\}, S - \{u\}]$ has the same color as $T[\{v_1, \dots, v_{k-1}\}]$ and uz is the opposite color. Therefore for $k \geq 3$ we have proven the fact.

If $k = 2$, then we need to verify that $z_1, z_2 \in Z$ have the same neighborhood in $\{u, v_1\}$. If they do not, then we may assume that $uz_1, v_1 z_2 \in EB(T)$ and $v_1 z_1, uz_2 \in EW(T)$. In this case, either $z_1 u v_1 z_2$ or $uz_1 z_2 v_1$ is a realization of P_4 depending on the color of $z_1 z_2$. Thus our proof is complete. \square

Fact 4.2.7. *Let T be a trigraph that is P_4 -induced-saturated. If $v_1 v_2 v_3$ is a gray K_3 and $x \notin \{v_1, v_2, v_3\}$ then either $x \sim v_1, v_2, v_3$ or $x \not\sim v_1, v_2, v_3$. That is a vertex not in a gray triangle must behave the same to the vertices in a gray triangle.*

Proof. Let T be a trigraph which is P_4 -induced-saturated and contains a gray triangle $v_1 v_2 v_3$. By Fact 4.2.1, and the symmetry of a gray triangle we only have to consider the case when

one edge is black. Without loss of generality, let $xv_1 \in EB(T)$, but $xv_2, xv_3 \in EW(T)$. As in Figure 4.4, T has an induced P_4 , $xv_1v_2v_3$. \square

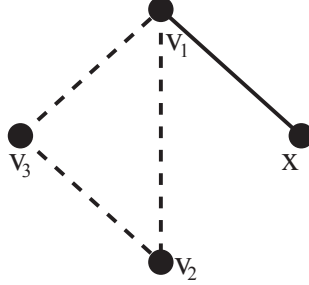


Figure 4.4: Example for proof of Fact 4.2.7. $xv_1v_2v_3$ is an induced P_4 in T .

Fact 4.2.8. *Let T be a trigraph which is P_4 -induced-saturated. If $V(T) = V_1 \cup V_2$ such that (V_1, V_2) is all white or all black then V_1 is P_4 -induced-saturated. Likewise for V_2 .*

Proof. Using Fact 4.2.1, we may assume that all edges between V_1 and V_2 are white. If V_1 is a gray complete graph we are done, so assume that V_1 is not a gray complete graph. Let e be a black or white edge in V_1 . Now, consider P an induced P_4 in $T \div e$. As there are only white edges between V_1 and V_2 , it must be that P is completely contained in V_1 . \square

Fact 4.2.9. *Let T be a trigraph which is P_4 -induced-saturated. If there exists a universal vertex $u \in T$, such that the components induced by $EB(T) \cup EG(T)$ of $T \setminus \{u\}$ are disconnected, then every component is P_4 -induced-saturated.*

Proof. Let us call the components induced by $EB(T) \cup EG(T)$ of $T \setminus \{u\}$, $C_1, C_2, C_3, \dots, C_k$ with $k \geq 1$. Let e be a black or white edge in C_1 . Let P be an induced P_4 in $T \div e$. We want to show that P can not contain u . To see this, we let v and w be two vertices in C_1 . We note that P can not have v and w adjacent and use u , otherwise u, v, w has $uw \in EB(T)$. Thus, it must be that P is of the form $vuwx$. However, x must be in C_1 since $wx \in EB(T) \cup EG(T)$. Thus, uwx has $ux \in EB(T)$. Therefore as desired, P can not contain u so it must be completely contained in C_1 . As there was nothing special about C_1 the same idea must hold in each C_i finishing the proof. \square

Fact 4.2.10. *Let T be a trigraph which is P_4 -induced-saturated. T can not contain an isolated vertex $y \in T$ and a vertex $u \in T$ which is universal to all vertices except y .*

Proof. We start by letting P be an induced P_4 in $T \div uy$. As uy was a white edge, P must use $uy \in EB(T)$. In addition, it must use two vertices in $V(T)$, say w and x , other than u and y , but $uw \in EB(T)$ and $ux \in EB(T)$. Thus, P can not be an induced P_4 . Therefore, we have shown that the edge uy is not induced-critical, so T is not P_4 -induced-saturated. \square

Fact 4.2.11. *Let T be a trigraph which is P_4 -induced-saturated. T can not contain a black bridge. That is T can not be 1-connected via a black edge.*

Proof. For sake of contradiction, let T be a trigraph that is P_4 -induced-saturated that does contain a black bridge say xy . We consider P an induced P_4 which occurs in $T \div xy$. If P uses the edge xy , then P would have existed in T . We let X be the component of $T \setminus \{xy\}$ which contains x and let Y be the component of $T \setminus \{xy\}$ which contains y . As P can not use the edge xy and xy was a bridge, P must be completely contained in either X or Y . Either way P existed in T which is the sought contradiction. \square

Fact 4.2.12. *Let T be a trigraph that is P_4 -induced-saturated. The black diameter of T is at most 2.*

Proof. If the diameter were 3 or more there must exist an induced P_4 . \square

The next fact is a very powerful fact. It is used to show that each equivalence class, created in the proof of the lower bound on the induced saturation number of P_4 , is P_4 -induced-saturated.

Fact 4.2.13. *Let T be a trigraph that is P_4 -induced-saturated. Let S be a subset of $V(T)$ such that (S, \bar{S}) has no gray edges. Let every pair of vertices in S have the same neighborhood in \bar{S} . Then S is P_4 -induced-saturated.*

Proof. Let uv be a black or white edge in S . Let P be an induced P_4 in $T \div uv$. We will show that the P can not contain vertices from S and \bar{S} .

We consider a few cases. The first case is that P can not have two neighbors, say $x, y \in \bar{S}$. That is $ux, uy, vx, vy \in EB(T)$. In this case, we have essentially two options for P , either it is $uxvy$, or it is $xuvy$. For each of these options P is not induced as $uy \in EB(T)$.

The second case is that P can not have two non-neighbors, say $x, y \in \overline{S}$. To see this we note that if there are two non-neighbors we can not create a P_4 .

The third case is that P can not have only one neighbor, say $x \in \overline{S}$. Let w be some other vertex in S . We have $ux, vy, wy \in EB(T)$. Now for P to exist we have essentially one option. P must be of the form $uvwx$, but $uw \in EB(T)$ so P is not induced.

The fourth case is that P can not have only one non-neighbor in \overline{S} . To see this we note that there is no edge from the non-neighbor to the other three vertices in P .

The fifth and final case is that P can not have one non-neighbor, $x \in \overline{S}$ and one neighbor, $y \in \overline{S}$. That is $ux, vx \in EW(T)$ and $uy, vy \in EB(T)$. In this case the only option for P is to be of the form $uvyx$, but $uy \in EB(T)$, so P is not induced.

□

Fact 4.2.13 could, in fact, be generalized to state the following:

Let T be a trigraph that is P_4 -induced-saturated. Let R and S be disjoint subsets of $V(T)$ such that (R, S) has no gray edges. Let every pair of vertices in S have the same neighborhood in R . Let e be a white or black edge in S , then any induced P_4 in $T \div e$ contains at most one member of R .

This generalization is proven by a case analysis similar to the above proof, but we did not need this amount of generality.

Fact 4.2.14. *Let T be a trigraph which is P_4 -induced-saturated. Let $u \in V(T)$. Let $V(T) = R \cup S$, where $R' = \{v \in V(T) : u \text{ and } v \text{ are adjacent via a gray pendant edge}\}$ and $R = R' \cup \{u\}$, and $S = V(T) \setminus R$. If u is black universal to all vertices in S and all edges between R and S are white, then S is P_4 -induced-saturated.*

Proof. Assume S is not a complete gray trigraph, that is there exists $w_1w_2 \in EB(T)$. Let P be an induced P_4 in $T \div w_1w_2$. If P contains the vertex u , then P must be of one of the following forms.

First, $P = w_1w_2ux$ with x any other vertex in T . This is not possible as $w_1u \in EB(T)$.

Second, $P = w_1uw_2x$, with x any other vertex in T . This too is not possible as $x \in S$ since $w_2x \in EG(T) \cup EB(T)$. Therefore, $ux \in EB(T)$ so P is not induced.

Third, $P = w_1uxy$, with x, y any other vertex in T . From our first option we know $x, y \notin S$, so it must be that $x, y \in R$. This is impossible since uv_1 and uv_2 were gray pendant edges.

Therefore we have shown that P can not use vertex u , so it is completely contained in S . \square

The final fact seems fairly obvious, but has shown to be hard to prove. It was instrumental for the proof of Theorem 3.3.1.

Fact 4.2.15. *If T is a P_4 -induced-saturated trigraph on at least four vertices, then T has a gray edge.*

Proof. We will prove this statement by strong induction.

Our base case is $V(T) = 4$. This is covered by Lemma 4.3.1.

We let n be a fixed, arbitrary, positive integer. For all integers m , with $4 \leq m < n$ a trigraph which is P_4 -induced-saturated with m vertices contains at least one gray edge. We will show that any P_4 -induced-saturated trigraph on n vertices contains a gray edge.

For sake of contradiction, assume T is a P_4 -induced-saturated trigraph on at least four vertices which does not contain a gray edge. We first note that there must be an induced P_3 in T , otherwise, T is disjoint cliques. If this were the case, Fact 4.2.8 and our assumption that T has no gray edges requires each clique to be at least size four. Once again, Lemma 4.3.1 requires that each clique actually has two gray edges.

Let the induced P_3 in T be uxv . We'll let the set X be the vertices that are adjacent to both u and v . The set Z_1 will be the vertices that are adjacent to only u . Likewise, the set Z_2 will be the vertices that are adjacent to only v . Finally, we'll let the set Y be all vertices that are not adjacent to u or v as in Figure 4.5.

We start the proof by trying to understand the structure of these sets and the relationships between these sets. Eventually we will show that there exists a universal vertex in our trigraph T , it will appear in X . We will then apply Fact 4.2.9 and induction to create our contradiction.

We first note that $x \in X$. Thus, we have that $X \neq \emptyset$.

Next we show that all edges between X and Z_1 as well as all edges between X and Z_2 must be black. To see this, let $x \in X, z_1 \in Z_1, z_2 \in Z_2$. Without loss of generality, assume $x_1z_1 \in EW(T)$.

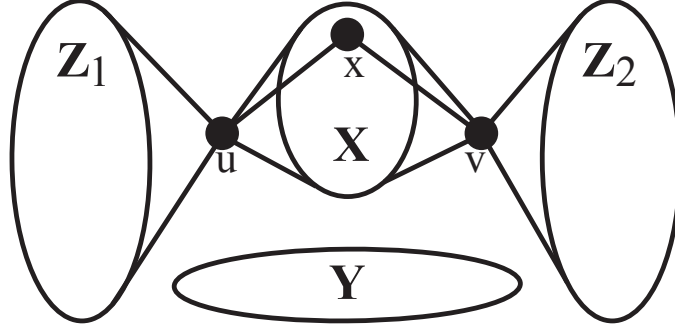


Figure 4.5: General set-up for proof of Fact 4.2.15. Trigraph T containing a P_3 with vertices uxv .

In this case, T contains an induced P_4 , z_1ux_1v contradicting that T is P_4 -induced-saturated. Likewise, without loss of generality, we may assume $xz_2 \in EW(T)$. In this case, T contains an induced P_4 , z_2vx_1u , again a contradiction.

We also show that all edges between Z_1 and Z_2 are white. Without loss of generality, we may assume $z_1z_2 \in EB(T)$. In this case, T contains an induced P_4 , uz_1z_uv , which is a contradiction.

We now shift our focus to the set Y . To gain a little more information about the structure of our trigraph T , we split the set Y . We let Y' be the vertices that are adjacent to some member of $Z_1 \cup Z_2$ and $Y'' = Y - Y'$.

We show that all edges between Y' and X are black. Let $y' \in Y'$ and $x_1 \in X$. Without loss of generality, let $y'z_1 \in EB(T)$ and $y'x_1 \in EW(T)$. In this case, T contains an induced P_4 , $y'z_1x_1v$, which is a contradiction to the fact that T is P_4 -induced-saturated.

We will slightly abuse terminology, we say a white component of some set of vertices is any subset of those vertices that can be reached via a white path. Now we consider Y'' . From Fact 4.2.12, we have that each $y'' \in Y''$ is adjacent to some $x_1 \in X$. We now show that every vertex in a white component of X has the same neighborhood in Y'' . Let $x_1, x_2 \in X$ and $y'' \in Y''$. Further, let $x_1y'' \in EB(T)$, $x_1x_2 \in EW(T)$, and $y''x_2 \in EW(T)$. This creates a contradiction, as T contains an induced P_4 with vertices $y''x_1ux_2$. Therefore, each $y'' \in Y''$ is adjacent to all or none of the vertices in the white components of X , which is what we hoped to show.

We step back a moment to see what we have proven thus far. Let X_1 be a white component of X . We know that $X_1 \sim Z_1, Z_2, Y'$ and $X - X_1$. Further, X_1 is either adjacent to Y'' or

is not adjacent to Y'' . Therefore, we may apply Fact 4.2.13 with $S = X_1$, to see that X_1 is P_4 -induced-saturated. As there is nothing special about X_1 , this is actually true of every white component in X . As we are assuming that T has no gray edges our inductive hypothesis states that each of these components is a single vertex. That is, X induces a complete black graph.

We are very close to our goal of showing that there exists a vertex in X which is universal. We need to focus a little more attention to the relationship between X and Y'' . Next, we show that all connected components of Y'' have the same neighborhood in X . Let $y_1'', y_2'' \in Y''$ such that $y_1'' y_2'' \in EB(T)$. Further, let $x_1 \in X$. Assume that $x_1 y_1'' \in EB(T)$, but $x_1 y_2'' \in EW(T)$. This is a contradiction as T contains an induced P_4 , $y_2'' y_1'' x_1 u$.

At last we are ready to state our claim.

Claim 4.2.16. *There exists a vertex $x' \in X$ which is adjacent via a black edge to every other vertex in T .*

Proof. From above we know that for each $x \in X$, we have $x \sim X, Z_1, Z_2$, and Y' . Therefore, it suffices to show that there exists a $x' \in X$ that is adjacent via a black edge to each vertex in Y'' .

Let $x' \in X$ have the largest black neighborhood in Y'' call this neighborhood Y_1'' . For sake of contradiction, we assume that there exists a $y'' \in Y'' - Y_1''$, that is $x' y'' \in EW(T)$. Once again by Fact 4.2.12, we know that each $y'' x_1 \in EB(T)$ for some $x_1 \in X$. Recalling from above that X induces a complete black graph we know $x_1 x' \in EB(T)$. Finally, we let $y_1'' \in Y_1''$, that is $x' y_1'' \in EB(T)$. Now, we recall from above that all connected components of Y'' have the same neighborhood in X . Using a variation of the contrapositive of this and the knowledge that $x' y'' \in EW(T)$ and $x' y_1 \in EB(T)$, we must have that $y'' y_1'' \in EW(T)$. To help clarify, Figure 4.6 shows the structure of the portion of T with which we are concerned.

Therefore, we only have the edge $x_1 y_1''$ left to worry about. We consider the two cases. First, if $x_1 y_1'' \in EW(T)$, then T contains an induced P_4 , $y'' x_1 x' y_1''$ which is a contradiction. Second, if $x_1 y_1'' \in EB(T)$, then $T[\{x_1\}, Y_1]$ is all black since all connected components of Y'' have the same neighborhood in X . This contradicts our choice of x' since x_1 then has a larger neighborhood in Y'' than x' . We have exhausted all possibilities, therefore our assumption that

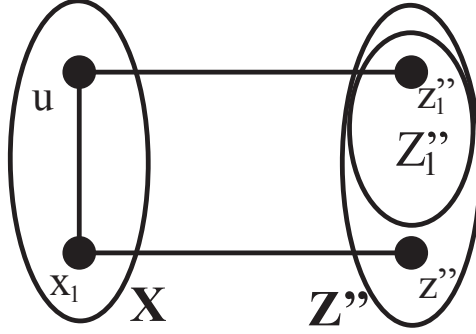


Figure 4.6: Figure for proof of Claim 4.2.16. Only the edge x_1y_1'' is left to be determined.

such a y'' exists must have been false so x' is in fact universal.

□

As desired, by Claim 4.2.16 there exists a universal vertex $x' \in X$. Thus, by Fact 4.2.9 we know $T - \{u\}$ has to be P_4 -induced-saturated. For notational convenience, let $M = T - \{u\}$.

As we have assumed that $|V(T)| \geq 4$, we must have that M is not empty. Thus, we may apply induction. Our inductive hypothesis requires that M contains a gray edge. Consequently, T must contain a gray edge which is a contradiction.

Therefore, our inductive step is complete and we have that every trigraph which is P_4 -induced saturated must contain a gray edge.

□

4.3 Base Cases

With the facts proven, we will now prove the base cases. We have stated and proven three base cases since our constructions depended on the modularity of n with respect to 3. Our first base case is $n = 4$.

Lemma 4.3.1. $\text{indsat}(4, P_4) = 2$.

Proof. We see that the graph given in Figure 4.7 is a trigraph on four vertices with $|EG(T)| = 2$ such that T is P_4 -induced-saturated. Thus, we have $\text{indsat}(4, P_4) \leq 2$.

Therefore, we need to show $\text{indsat}(4, P_4) \geq 2$. For sake of contradiction, assume $\text{indsat}(4, P_4) < 2$. Let T be a trigraph which is P_4 -induced-saturated and $|EG(T)| < 2$. We note

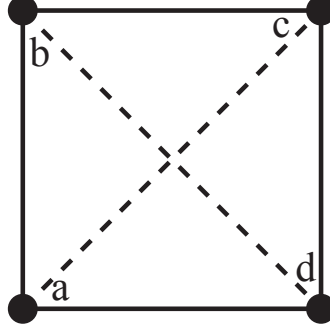


Figure 4.7: A trigraph on four vertices which is P_4 -induced-saturated.

that this implies that there are at least two vertices in T which are not incident to a gray edge. Now, from Fact 4.2.3 we know that T does not have a pendant edge. Further, by Fact 4.2.1 we know that \overline{T} does not have a pendant edge. Combining this information, we have that each of the at least two vertices that aren't incident to a gray edge must have degree 0 or 3. As these at least two vertices are either adjacent or not, they are either both degree 0 or both degree 3. Without loss of generality using Fact 4.2.1, let them have degree 0. Thus, our trigraph T must look something like the trigraph shown in Figure 4.8. That is T has four vertices a, b, w_1, w_2 , one gray edge w_1w_2 , and all other edges white. Now, we consider P an induced P_4 in $T \div ab$. We see that P can not exist since $T \div ab$ has only two non-white edges. Hence, we have that T is not P_4 -induced-saturated, the desired contradiction. \square

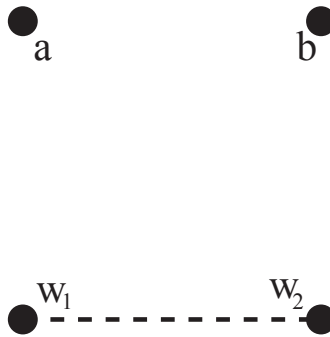


Figure 4.8: Example for base case $n = 4$. Edge ab is not induced-critical.

With the base case $n = 4$ completed. Our next base case is $n = 5$.

Lemma 4.3.2. $indsat(5, P_4) = 2$.

Proof. We see that the graph given in Figure 4.9 is a trigraph on five vertices with $|EG(T)| = 2$ such that T is P_4 -induced-saturated. Thus, we have $\text{indsat}(5, P_4) \leq 2$.

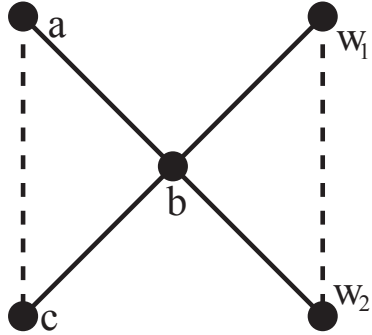


Figure 4.9: A trigraph on five vertices which is P_4 -induced-saturated.

Therefore, we need to show $\text{indsat}(5, P_4) \geq 2$. For sake of contradiction, assume $\text{indsat}(5, P_4) < 2$. Let T be a trigraph which is P_4 -induced-saturated and $|EG(T)| < 2$. We note that this implies that there are at least three vertices in T which are not incident to a gray edge. Now, from Fact 4.2.3 we know that T does not have a pendant edge. Further, by Fact 4.2.1 we know that \overline{T} does not have a pendant edge. Combining this information, we have that each of the at least three vertices that aren't incident to a gray edge must have degree 0, 2, or 4. Without loss of generality, we can assume that none of them are degree 4. Thus we have a few cases to consider. These cases are organized by the degree sequence of the three vertices which are not incident to a gray edge.

Case 1 : $\{(0, 0, 0)\}$. Our notation $(0, 0, 0)$ means the three vertices that aren't adjacent to the gray edge have degree 0. Let T be a trigraph with vertices a, b, c, w_1 , and w_2 , and the only edge is the gray edge w_1w_2 , as in Figure 4.16a. In this case we can see that $T \div ab$ does not contain an induced P_4 as $T \div ab$ has only two non-white edges.

Case 2 : $\{(0, 0, 2)\}$. As in Figure 4.16b, we can see the edge ab is not induced-critical.

Case 3a : $\{(0, 2, 2)\}$. As in Figure 4.16c, we can see the edge ab is not induced-critical.

Case 3b : $\{(0, 2, 2)\}$. As in Figure 4.16d, we see that w_1abw_2 is an induced P_4 in T .

Case 4a : $\{(2, 2, 2)\}$. As in Figure 4.16e, we can see the edge ab is not induced-critical.

Case 4b : $\{(2, 2, 2)\}$. As in Figure 4.16f, we see that w_1abc is an induced P_4 in T .

Case 4c : $\{(2, 2, 2)\}$. As in Figure 4.16g, we can see the edge ab is not induced-critical.

Case 4d : $\{(2, 2, 2)\}$. As in Figure 4.16h, we see that bw_1cw_2 is an induced P_4 in T .

Case 4e : $\{(2, 2, 2)\}$. As in Figure 4.16i, we see that aw_1cw_2 is an induced P_4 in T . \square

With the base cases $n = 4, 5$ complete. We have our final base case $n = 6$.

Lemma 4.3.3. $indsat(6, P_4) = 3$.

Proof. We see that the graph given in Figure 4.10 is a graph on six vertices with $|EG(G)| = 3$ such that G is P_4 -induced-saturated. Thus, we have $indsat(6, P_4) \leq 3$.

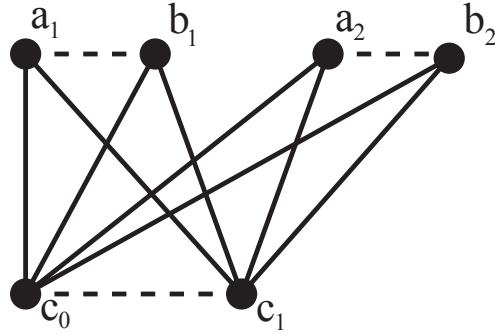


Figure 4.10: A trigraph on six vertices which is P_4 -induced-saturated.

Therefore, we need to show $indsat(6, P_4) \geq 3$. For sake of contradiction, assume $indsat(6, P_4) < 3$. Let T be a trigraph which is P_4 -induced-saturated and $|EG(T)| < 3$. We note that this implies that there are at least two vertices in T which are not incident to a gray edge. Now, from Fact 4.2.3 we know that T does not have a pendant edge. Further, by Fact 4.2.1 we know that \overline{T} does not have a pendant edge. Combining this information, we have that each of the at least two vertices that aren't incident to a gray edge must have degree 0, 2, 3, or 5. Without loss of generality, we can assume that none of them are degree 5. We have three cases to consider; first the edges form a star; second the gray edges are not incident and have no black edges between them; third the gray edges are not incident and have black edges between them. Each case will have multiple subcases.

Case 1 : {The two gray edges form a star}. The two gray edges in T are incident forming a gray star with center u and leaves v_1 and v_2 , see any figures in Figure 4.11. Using Fact 4.2.1 we may assume $v_1v_2 \in EW(T)$. Thus, we only have a few subcases to consider.

Case 1a : {No non-leaves are adjacent to the center}. The first subcase is that none of w_1, w_2 , or w_3 is adjacent to u via a black edge, see Figure 4.11a. Thus, by Fact 4.2.5, $T[\{w_1, w_2, w_3\}, \{u, v_1, v_2\}]$ is all white edges. Thus, Fact 4.2.8 forces $T[w_1, w_2, w_3]$ to be a gray triangle which is a contradiction to our assumption that T has only two gray edges.

Case 1b : {One non-leaf is adjacent to the center}. The second subcase is that one of w_1, w_2 , or w_3 is adjacent to u via a black edge, see Figure 4.11b. Let $uw_1 \in EB(T)$, but $uw_2 \in EW(T)$ and $uw_3 \in EW(T)$. By Fact 4.2.3 and Fact 4.2.8 we know that $T[\{w_1, w_2, w_3\}]$ is a black triangle. This is because Fact 4.2.5 guarantees that w_2 and w_3 can only have black edges to each other and w_1 . Now as $w_1v_1 \in EW(T)$, $w_3w_1uv_1$ is an induced P_4 in T . Therefore we have our sought contradiction.

Case 1c : {Two non-leaves are adjacent to the center}. The third subcase that two of w_1, w_2 , or w_3 is adjacent to u via a black edge, see Figure 4.11c. Let $uw_1 \in EB(T)$ and $uw_2 \in EB(T)$, but $uw_3 \in EW(T)$. By Fact 4.2.3 and Fact 4.2.8 we know that $w_1w_3, w_2w_3 \in EB(T)$. This is because Fact 4.2.5 guarantees that w_2 and w_3 can only have black edges to each other and w_1 . Now as $w_1v_1 \in EW(T)$, $w_3w_1uv_1$ is an induced P_4 in T . Therefore we have our sought contradiction.

Case 1d : {All three non-leaves are adjacent to the center}. The final subcase is that all of w_1, w_2 and w_3 are adjacent to u via a black edge, see Figure 4.11d. Let $uw_1, uw_2, uw_3 \in EB(T)$. Fact 4.2.14 forces $T[w_1, w_2, w_3]$ to be a gray triangle, which once again contradicts the assumption that T has only two gray edges.

We have exhausted all possibilities for the two gray edges being incident. We now focus on our second case. The two gray edges are not incident and there are no black edges between them.

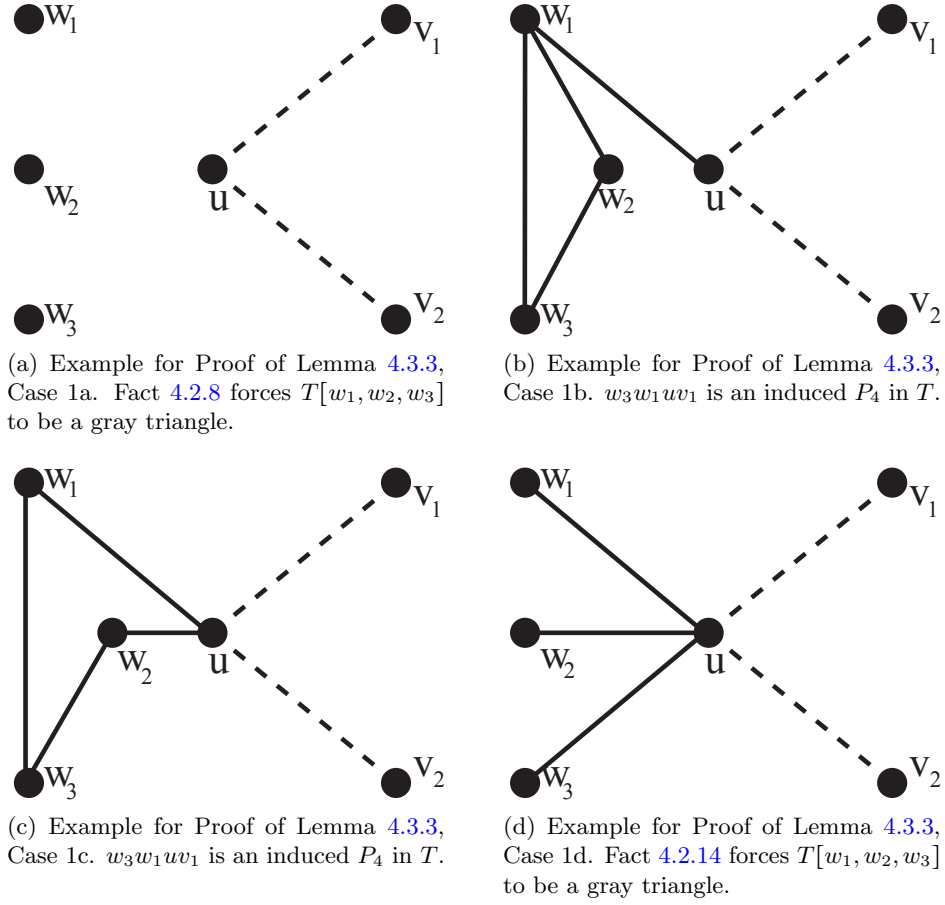


Figure 4.11: Trigraphs on six vertices with two incident gray edges.

Case 2 : {The two gray edges are not incident and no black edges between them}.

In this case, we will consider our trigraph to have two gray edges uv_3 and v_1v_2 . Let the remaining two vertices be w_1 and w_2 , with $\deg(w_1) \geq \deg(w_2)$. We're assuming the gray edges in G are not incident and there are no black edges between them, an example is shown in Figure 4.12.

We start this case by proving a very useful claim.

Claim 4.3.4. *If the degree of w_1 is at least 2, then w_1 must be adjacent to both vertices of a gray edge.*

Proof. Assume not, that is without loss of generality assume $w_1w_2, v_1w_1 \in EB(T)$. We note

that if $v_1w_2 \in EB(T)$ then by Fact 4.2.13 with $S = \{w_1, w_2\}$, we have that $w_1w_2 \in EG(T)$, a contradiction. However, if $v_1w_2 \in EW(T)$ then $w_2w_1v_1v_2$ is an induced P_4 in T . Therefore we must have that $w_1v_2 \in EB(T)$. \square

Case 2a : {The two vertices not incident to a gray edge are connected}. We consider the cases when $w_1w_2 \in EB(T)$. Recall that our trigraph T has 6 vertices, containing 2 gray edges, with w_1, w_2 not incident to a gray edge. By Fact 4.2.3, we know that the degree of w_1 is at least two. Thus by Claim 4.3.4, we must have that the degree of w_1 is at least three. Recalling that the degree of w_1 can't be four, it must be that the degree of w_1 is three or five. We consider each of these as subcases now.

Subcase i : {Highest degree vertex not incident to a gray edge has degree three}.

The vertex w_1 has degree three. Without loss of generality using Claim 4.3.4, we let $w_1w_2, w_1v_1, w_1v_2 \in EB(T)$. We start by showing that $v_2w_2 \in EW(T)$. If not, we apply Claim 4.3.4, to see that $v_2w_2, v_1w_2 \in EB(T)$. However by Fact 4.2.13 with $S = \{w_1, w_2\}$, we have that $w_1w_2 \in EG(T)$, a contradiction. Next we observe that $w_2u \in EW(T)$, otherwise $v_2w_1w_2u$ is an induced P_4 in T . Once again, we use Claim 4.3.4, to see that $w_2v_3 \in EW(T)$. Therefore we must have that w_2 is a pendant edge as in Figure 4.12, but this is a contradiction to Fact 4.2.3. Hence, w_1 can not have degree three.

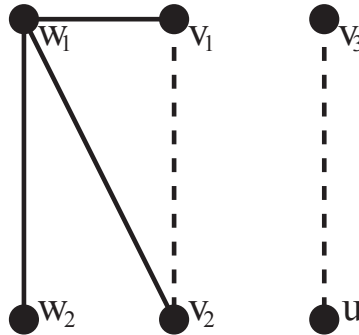


Figure 4.12: Example for proof of Lemma 4.3.3, Case 2a, Subcase i. Vertex w_1 has degree 3.

Subcase ii : {The highest degree vertex not incident to a gray edge has degree five}. The vertex w_1 has degree five. We show that w_2 can not also have degree five. If so, both w_1 and w_2 are universal vertices. We show that the edge w_1w_2 is not induced-critical. Let P be an induced P_4 in $T \setminus w_1w_2$. It must be that P contains both w_1 and w_2 . Thus, it must be of the form w_1xw_2y or w_1xyw_2 , but in either case $w_1y \in EB(T)$ so P is not an induced P_4 . With this case eliminated, we have two other cases to consider. Let $S_1 = \{v_1, v_2\}$ and $S_2 = \{v_3, u\}$. If w_2 is adjacent to vertices only in S_1 , then we use Fact 4.2.9 with universal vertex w_1 to see that $w_2 \cup S_1$ and S_2 must be gray complete graphs, a contradiction. The other option is that w_2 is adjacent to at least one vertex in each of S_1 and S_2 . In this case the trigraph T will contain an induced P_4 . As an illustration say $w_2v_2 \in EB(T)$ and $w_2v_3 \in EB(T)$ then $v_1v_2w_2v_3$ is an induced P_4 in T , where we assume $w_2v_1 \in EW(T)$ since the degree of w_2 is not 5.

Case 2b : {The two vertices not incident to a gray edge are not connected}. We consider the cases when $w_1w_2 \in EW(T)$. Recall that our trigraph T has 6 vertices, containing 2 gray edges, with w_1, w_2 not incident to a gray edge, and the degree of w_2 has to be either 0, 2, 3, or 5. If $w_1w_2 \in EW(T)$ then we have eliminated the case when the degree of w_2 is 5. Next, we show that the degree of w_1 can not be three. Without loss of generality, we may assume $w_2v_1, w_2v_2, w_2v_3 \in EB(T)$, but $w_2u \in EW(T)$. So T contains the induced P_4 , $uv_3w_2v_1$.

We use Claim 4.3.4, to note that if the degree of w_2 is 2, then both edges must be incident to vertices of the same gray edge, as shown in any of the graph in Figure 4.13. We have several subcases to consider.

Subcase i : {The two vertices not incident to a gray edge have degree 0}. First we consider the case that both w_1 and w_2 have degree 0, as in Figure 4.13a. We use Fact 4.2.8 to see that $w_1w_2 \in EG(T)$, a contradiction.

Subcase ii : {The two vertices not incident to a gray edge have degree 0 and

2}. Second, we consider the case that the degree of w_1 is 2 and the degree of w_2 is 0. Without loss of generality we let $w_1v_1, w_1v_2 \in EB(T)$, as in Figure 4.13b. Again, Fact 4.2.8 tells us that $T[w_1, v_1, v_2]$ must be a gray triangle, a contradiction.

Subcase iii : {The two vertices not incident to a gray edge have degree 2 and common neighbors}. Third, we consider the case that the degree of w_1 and w_2 is 2 and they are adjacent to the same two vertices. Without loss of generality, we let $w_1v_1, w_1v_2, w_2v_1, w_2v_2 \in EB(T)$, as in Figure 4.13c. We see that $w_1w_2 \in EG(T)$ by Fact 4.2.13 with $S = \{w_1, w_2\}$, a contradiction.

Subcase iv : {The two vertices not incident to a gray edge have degree 2 and different neighbors}. Finally we consider the case that the degree of w_1 and w_2 is 2 and they are adjacent to different vertices. Without loss of generality, we let $w_1v_1, w_1v_2, w_2v_3, w_2v_4 \in EB(T)$, as in Figure 4.13d. By Fact 4.2.8, we have that $T[w_1, v_1, v_2]$ must be a complete gray graph, but this contradicts our assumption that $|EG(T)| \leq 2$.

Hence we have completed Case 2, that is we have shown that the two gray edges can not be disjoint and have only white edges between them. Therefore, we have shown that we can not have the two gray edges which are either incident or which are disjoint with only white edges between them. Thus, we only have to consider the case when the two gray edges are disjoint with some black edges between them.

Case 3 : {The two gray edges are not incident and have black edges between them}.

Let $v_1v_2 \in EG(T)$ and $v_3v_4 \in EG(T)$. We start by showing that there is only one way that it is possible to have black edges between any of these vertices. By Fact 4.2.1, we know that we only have to consider the cases of having 0,1, or 2 black edges. The case of 0 black edges was covered in Case 2 of this proof. We show now that if there were a single black edge between the two gray edges there would be an induced P_4 . Without loss of generality, say $v_1v_3 \in EB(T)$,

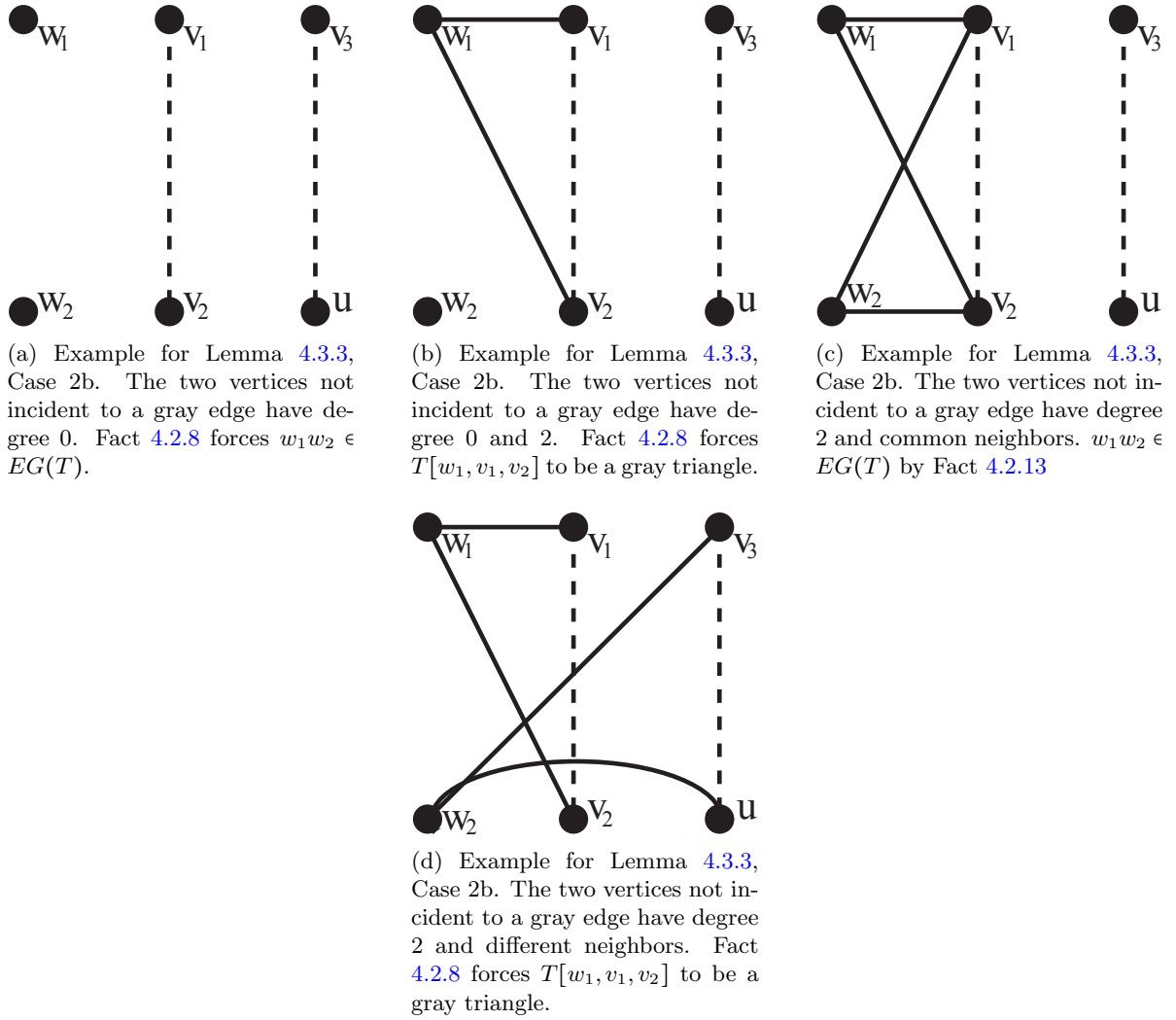


Figure 4.13: Trigraphs on six vertices with $w_1w_2 \in EW(T)$ and each of w_1 and w_2 having degree 0 or 2.

then $v_2v_1uv_3$ is an induced P_4 in T . Therefore, we must have two black edges between the two gray edges. We show that if these black edges are not incident, then we once again have an induced P_4 . Without loss of generality let $v_2u, v_1v_3 \in EB(T)$, then $v_2uv_3v_1$ is an induced P_4 in T . Therefore, the only case left to consider is when the two black edges are incident. We'll let $uv_1, uv_2 \in EB(T)$. Further we will let w_1 and w_2 be the vertices that are not adjacent to a gray edge. Recall that by Fact 4.2.1 and Fact 4.2.3 the degree of w_1 and w_2 must be either 0, 2, 3, or 5. A generic example of what T can look like is provided in Figure 4.14.

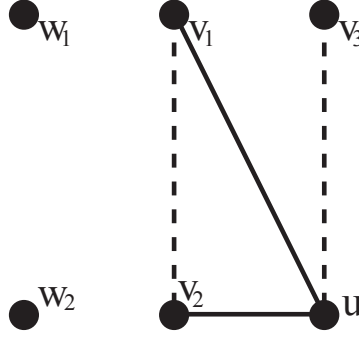


Figure 4.14: General idea for Case 3.

We now prove a couple claims that will be useful for the proof.

Claim 4.3.5. *It must be that $w_1v_3 \in EW(T)$ and $w_2v_3 \in EW(T)$.*

Proof. Without loss of generality, for sake of contradiction let $w_1v_3 \in EB(T)$. We have two cases to consider. First if $w_1v_2 \in EW(T)$, then $w_1v_3uv_2$ is an induced P_4 in T . Second, if $w_1v_2 \in EB(T)$, then $uv_2w_1v_3$ is an induced P_4 in T . In either case, we achieve our desired contradiction. \square

Claim 4.3.6. *If $w_1u \in EW(T)$, then $w_1v_i \in EW(T)$ for $1 \leq i \leq 3$.*

Proof. Without loss of generality, for sake of contradiction let $w_1v_1 \in EB(T)$ then $w_1v_1uv_3$ is an induced P_4 in T since $w_1v_3 \in EW(T)$ by Claim 4.3.5. This is our desired contradiction. \square

We show now that we can not have the degree of w_1 or the degree of w_2 be 0. Without loss of generality, let the degree of w_1 be 0. Consider P an induced P_4 in the trigraph $T \div w_1u$. P must contain the edge w_1u . Without loss of generality, since w_1 has only one non-white edge, we'll assume P is of the form w_1uxy , where x and y are two other vertices in T . A quick exhaustive search shows that we can not have $x = v_i$, $y = v_j$ for $1 \leq i \neq j \leq 3$ as the edges v_1v_3 and v_2v_3 don't exist and we can't use v_1v_2 because both vertices are adjacent to u . Therefore, we may assume that either $x = w_2$ or $y = w_2$.

By the contrapositive of Claim 4.3.6 applied to vertex w_2 , if $w_2v_1 \in EB(T)$ or $w_2v_2 \in EB(T)$, then $uw_2 \in EB(T)$. Therefore, we must either have $x = v_3$ and $y = w_2$ or $x = w_2$. By Claim

4.3.5, we know that we can not have $x = v_3$ and $y = w_2$, as $v_3w_2 \in EW(T)$. Thus, our only remaining option with degree of w_1 to be 0 is that $x = w_2$. If $x = w_2$, then y must be either v_1 or v_2 , which is a contradiction since $uv_1, uv_2 \in EB(T)$.

Thus the degree of w_1, w_2 is either 2 or 3. The contrapositive of Claim 4.3.6 forces $uw_1, uw_2 \in EB(T)$, as in Figure 4.15. We have now that u is universal to every vertex except v_3 , but uv_3 is a gray pendant edge. Therefore, by Fact 4.2.14, we must have that $T[v_1, v_2, w_1, w_2]$ is P_4 -induced-saturated. However, this is a contradiction to Lemma 4.3.1 since there is only one gray edge induced by these four vertices.

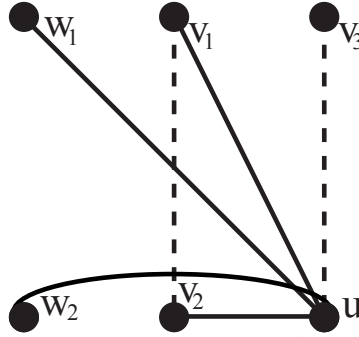


Figure 4.15: Example for Case 3 in the proof of Lemma 4.3.3. A trigraph on 6 vertices with two edges between the two gray edges.

□

We have now proven the facts and base cases. Therefore, we have completely proven that $\text{indsat}(n, P_4) = \left\lceil \frac{n+1}{3} \right\rceil$.

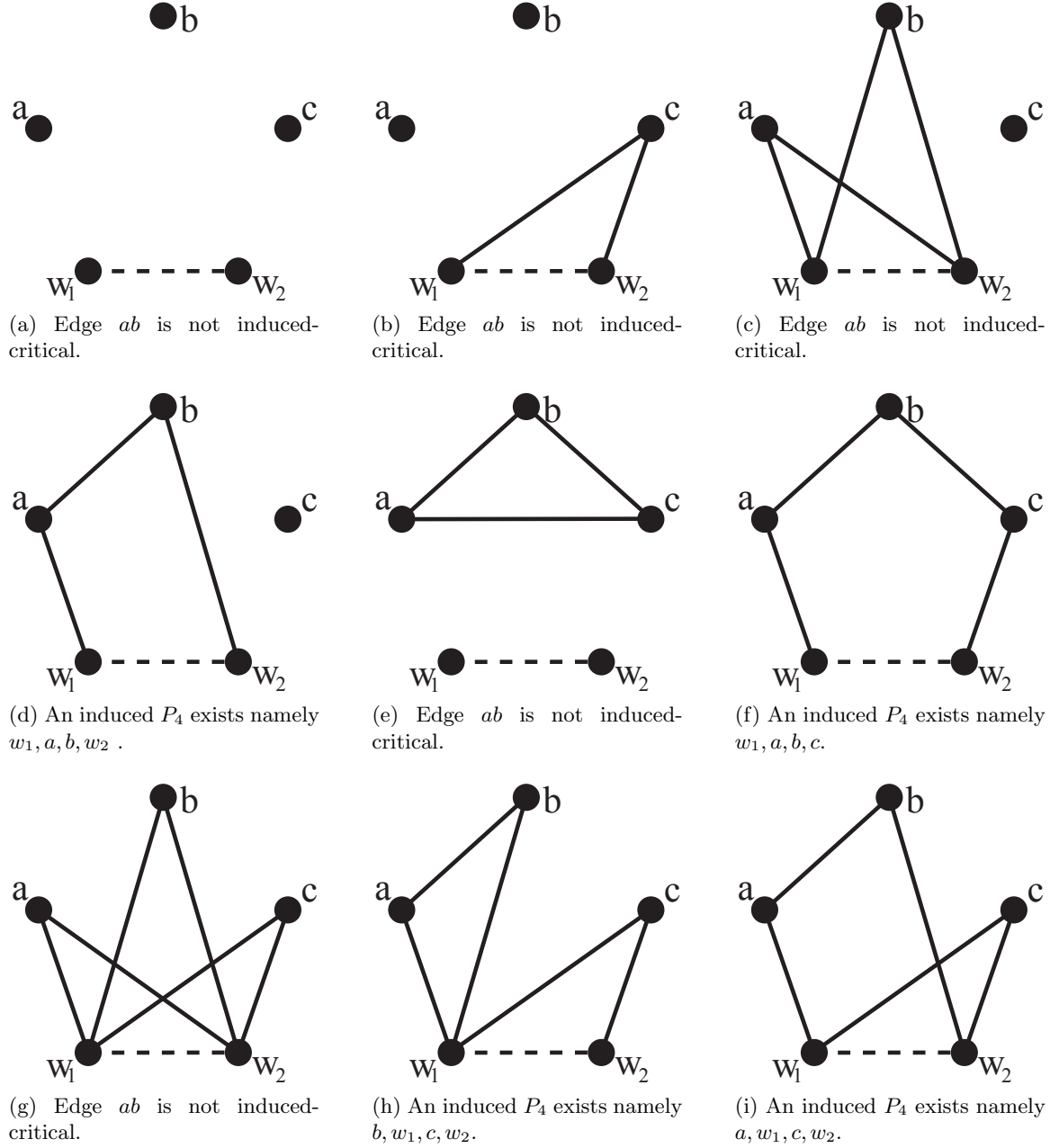


Figure 4.16: Trigraphs on five vertices with all cases for vertices not incident to one of the two gray edges having degree zero or two.

CHAPTER 5. INJECTIVE COLORINGS RESULT

In this chapter we will introduce the notion of injective coloring and then generalize the notion of injective coloring. We will present a nice result on this generalization of the injective coloring number.

5.1 Background and Definitions

In this section we will provide a brief literature review and some background definitions. The literature review is very brief because at this point injective colorings are still a very new topic having been introduced in 2002.

5.1.1 Literature Review

In [22], Hahn et al. introduce the concept of an injective coloring. They defined it by saying that a function that colors the vertices of a graph is said to be injective if the restriction of this function to the neighborhood of any vertex is itself an injective map. In other words, the neighborhood of a vertex should be totally multicolored. The injective chromatic number $\chi_i(G)$ of a graph G is the least k such that there is an injective k -coloring. They also related the injective chromatic number to the code covering number of the hypercube and established some nice bounds on the injective chromatic number. We mention two such bounds here as we'll use them later. First, if G is a graph with maximum degree $\Delta(G)$, then $\chi_i(G) \geq \Delta(G)$. Second, if G is a graph with maximum degree $\Delta(G)$, then $\chi_i(G) \leq \Delta(\Delta - 1) + 1$.

After Hahn et al., a flourish of activity came about. The focus quickly narrowed to graphs that were somehow restricted to be sparse. In [10] and [11], Cranston, Kim, and Yu also Doyon, Hahn, and Raspaud in [12] found better bounds on the injective chromatic number for graphs

that had constraints on the maximum average degree. In [26], Lužar, Škrekovski, and Tancer worked on improving the known bounds on the injective chromatic number for planar graphs.

5.1.2 Definitions

Following the notation and definitions given in the literature, we make a series of definitions to generalize injective coloring.

We start by letting G be a simple graph, say $G = (V, E)$. Following West, we define a path as a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. Also, the length of a path is the number of edges. A k -coloring of a graph G is a coloring $c : V(G) \rightarrow [k]$.

We define a k -coloring on a graph G to be **j -proper** if for any two vertices $u \neq v \in V(G)$ for which there exists a u, v -path of length j , u and v receive different labels. Note here that we are not concerned about paths of any length other than j .

A graph is **j -injective k -colorable** if it has a j -proper k -coloring. The **j -injective chromatic number**, ${}_j\chi_i(G)$, is the least k such that G is j -injective k -colorable. These definitions follow the concepts of chromatic number and injective chromatic number of a graph. In fact, using our definition the chromatic number would be the 1-injective chromatic number and the injective chromatic number would be the 2-injective chromatic number.

At this point, it is necessary to show that ${}_j\chi_i(G)$ is well-defined for all integers j with $j \geq 1$. To see this let G be a simple, undirected graph with $G = (V, E)$, $|V| = n$, $u \neq w \in V$, and j an integer with $j \geq 1$. We consider a rainbow coloring of G , that is we assign each vertex in V a different color. Then there are only two cases. First, there does not exist a u, w -path in G of length j in which case we don't care about the coloring of u and w . Second, there does exist a u, w -path in G of length j . In this case, we need u and w to have different colors, but since G is rainbow colored they do. Therefore, every graph G has a j -proper $|V(G)|$ -coloring. Thus, the j -injective chromatic number of G is well defined.

Next we define a collection of different graphs. We will use these graphs to create some relationships. In this section, to ease our notation, we will use the convention of referring to the i^{th} component of a vector \mathbf{v} as $\mathbf{v}(i)$. We say that the hamming distance between two

vectors \mathbf{v} and \mathbf{w} is the number of components in which \mathbf{v} and \mathbf{w} do not agree. We will use the standard notation for Hamming distance, $H(\mathbf{v}, \mathbf{w})$. We define the n -cube, Q_n , to be the graph on the vertex set $\{0, 1\}^n$ with $\{\mathbf{a}, \mathbf{b}\} \in E(Q_n)$ if and only if $H(\mathbf{a}, \mathbf{b}) = 1$. We let ${}_kH_n$ be the graph on the vertex set $\{0, 1\}^n$ with an even number of 1's with $\{\mathbf{a}, \mathbf{b}\} \in E({}_kH_n)$ if and only if $H(\mathbf{a}, \mathbf{b}) = 2i$ for some i , with $1 \leq i \leq k \leq \lfloor \frac{n}{2} \rfloor$. We note that ${}_kH_n \cong K_{n-1}$ if $k \geq \lceil \frac{n}{2} \rceil$. We define $G^{(k)}$ by $V(G^{(k)}) = V(G)$ and $E(G^{(k)}) = \{\{x, y\} : \text{there is a path of length } k \text{ joining } x \text{ to } y\}$, note that with this notation $G^{(1)}$ is the same as G . Similarly, we define $G^k = \bigcup_{i=1}^k G^{(i)}$.

Finally we define a code and the code covering number, which ultimately we will relate to the j -injective chromatic number.

A code S is a subset of the vertices of (Q_n) . We say that S is k -error-correcting if the Hamming distance of any two distinct vertices of S is at least $2k + 1$. The code covering number $\gamma_k(Q_n)$ is the minimum number t of k -error-correcting codes, S_1, S_2, \dots, S_t such that $V(Q_n) = \bigcup_{i=1}^t S_i$.

5.2 Main Result

We start this section by mentioning an open question that we worked on, but were not able to solve. After that we'll show that there is not necessarily a relationship between varying levels of j -injective chromatic number, finally we follow the ideas of Hahn et al. and show that the j -injective chromatic number has a nice relationship with code cover numbers.

5.2.1 Remaining Open Question

We started studying the injective chromatic number via an open question, 'What is the injective chromatic number of a 3-regular planar graph?'. It seemed to be a fairly straightforward and easily solved question. There were known bounds, the generic upper bound given by Hahn et al. of $\chi_i(G) \leq 7 = 3(3 - 1) + 1$ as a 3-regular graph has $\Delta(G) = 3$. There was also a known lower bound of 5 by construction, which we have shown in Figure 5.1. In this figure we provided an injective 5 coloring. We will leave it to the interested reader to establish that it is not possible to find a proper injective 4 coloring. We spent a lot of time working on this open

question, but to no avail. Some of the proof techniques that we attempted without success were induction, discharging, and an algebraic method.

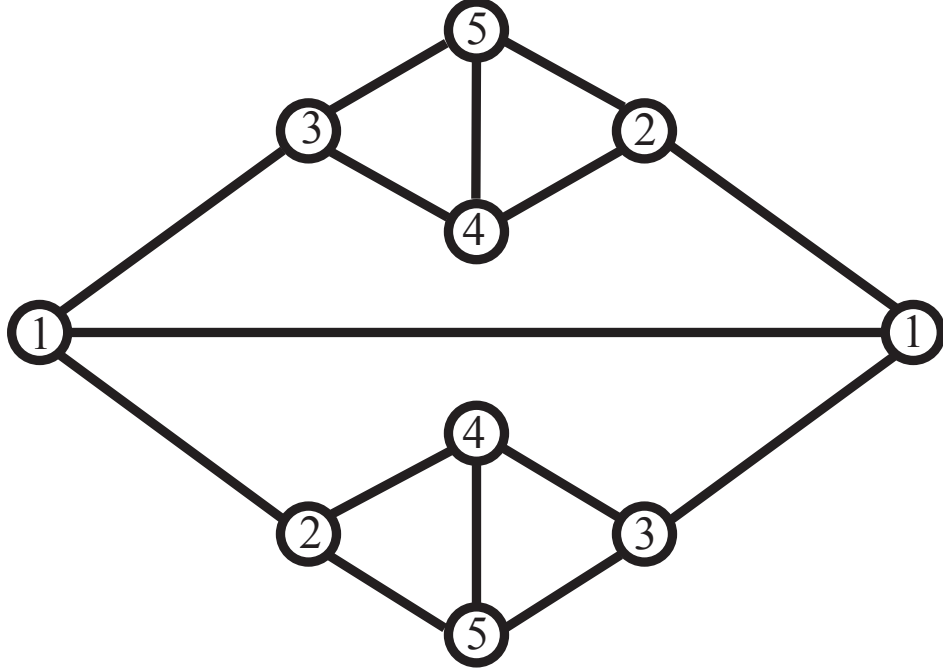


Figure 5.1: A three regular planar graph with a proper injective 5 coloring.

5.2.2 No Relationship for Varying Levels of j

In this subsection we will show that there is not necessarily a relationship between the j -injective chromatic number and the $(j+1)$ -injective chromatic number on a graph G for any $j \geq 2$.

Proposition 5.2.1. *Let G and H be graphs with n vertices. It is possible for ${}_j\chi_i(G) < {}_{j+1}\chi_i(G)$ and ${}_j\chi_i(H) > {}_{j+1}\chi_i(H)$.*

Proof. We will prove this proposition with a couple of examples that can easily be generalized. The graph in Figure 5.2, shows that $\chi_i(G) > {}_4\chi_i(G)$. In Figure 5.2a a 2-proper injective 3-coloring of G is shown. Since $\Delta(G) = 3$, using the bound from Hahn et al. we have that $\chi_i(G) = 3$. Likewise since a 4-proper injective 2-coloring is given in Figure 5.2b we know ${}_4\chi_i(G) < 3$. This example can be generalized to ${}_{2n}\chi_i(G)$ by replacing the two $K_{2,2}$ s with $K_{n,n}$ s.

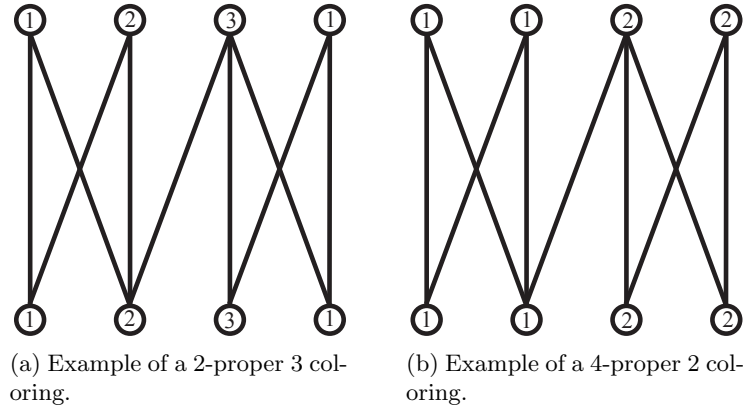


Figure 5.2: A graph G with injective colorings for different values of j .

Similar to above, the graph in Figure 5.3, shows that $\chi_i(H) > {}_4\chi_i(H)$. In Figure 5.3a a 2-proper injective 3-coloring of H is shown. Since $\Delta(G) = 3$, we have that $\chi_i(G) = 3$. A simple inspection shows that the 4-injective 4-coloring given in Figure 5.3b uses the fewest number of possible colors, since all the vertices on the top half of the graph have paths of length 4 between them. Again, this example can be generalized to ${}_{2n}\chi_i(H)$ by extending the graph to contain $2n$ vertices. \square

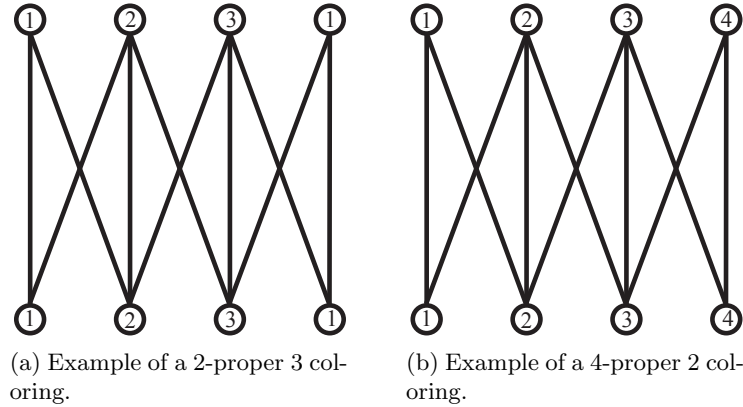


Figure 5.3: A graph G with injective colorings for different values of j .

This was an unexpected and disheartening result. We had hope that for higher values of j the j -injective chromatic number would decrease, since our intuition led us to believe that there would be fewer paths of longer length. We still believe that given a generic graph, the j -injective chromatic number will decrease as j increases.

5.2.3 Relationship of j -injective Chromatic Number to Code Cover Number

Theorem 5.2.2. $_{2k}\chi_i(Q_{n+1}) = \gamma_k(Q_n)$, for $k, n \in \mathbb{N}$ with $k \leq \frac{n-1}{2}$.

We note that when $k = 2$ this is equivalent to the statement of Lemma 11 in [22], by Hahn et al.

Proof. Following the ideas presented in [22], we know that $_{2k}\chi_i(Q_{n+1}) = \chi(Q_{n+1}^{(2k)})$ by definition of $\chi(Q_{n+1}^{(2k)})$. Then, $\chi(Q_{n+1}^{(2k)}) = \chi({}_kH_{n+1})$ since $Q_{n+1}^{(2k)}$ is two disjoint copies of ${}_kH_{n+1}$ by Lemma 5.2.3. Finally, we have that $\chi({}_kH_{n+1}) = \chi(Q_n^{2k})$, since ${}_kH_{n+1} \cong Q_n^{2k}$ by Lemma 5.2.4. Now, the color classes of any proper coloring of Q_n^{2k} are k -error-correcting codes. Vice versa, any decomposition of the vertex set of the n -cube into k -error-correcting codes yields a k -proper coloring of Q_n^{2k} . Thus, it follows that $_{2k}\chi_i(Q_{n+1}) = \gamma_k(Q_n)$. \square

Lemma 5.2.3. $Q_n^{(2k)}$ has two components both isomorphic to ${}_kH_n$, for $k \leq \frac{n-1}{2}$.

Proof. First, we will show that the two components correspond to the vectors with an even number of 1's and the vectors with an odd number of 1's. This is clear since if we start with a vector with an even number of 1's and flip two components we still have a vector with an even number of 1's. Likewise, flipping two components in a vector with an odd number of 1's results in a vector with an odd number of 1's.

Next we will show that each component is isomorphic to ${}_kH_n$. To show one direction, we will consider the set of vectors, V , with an even number of 1's. Let $\mathbf{v}, \mathbf{w} \in V$ such that \mathbf{v} is adjacent to \mathbf{w} in $Q_n^{(2k)}$; that is there is a path of length $2k$ between them. Therefore, by definition it is clear that \mathbf{v} is adjacent to \mathbf{w} in ${}_kH_n$. To show the other direction, let $\mathbf{v}, \mathbf{w} \in V$ such that \mathbf{v} is adjacent to \mathbf{w} in ${}_kH_n$; that is there is a path of length $2i$ between them for some i , with $1 \leq i \leq k \leq \lfloor \frac{n}{2} \rfloor$. Now, it suffices to show that for $Q_n^{(2k)}$ if there exists a path of length j , with $j < 2k \leq n-1$ from vertex \mathbf{v} to \mathbf{w} , then there also exists a path of length $j+2$. Since $j \leq n-1$ there is one component of \mathbf{v} and \mathbf{w} which is the same, say this is component 1. Thus, there exists \mathbf{b} and \mathbf{c} , two adjacent vertices on the path from \mathbf{u} to \mathbf{v} , with $\mathbf{b}(i) = \mathbf{c}(i)$ for all i , except say $i = 2$. Now we can divert the path to go from \mathbf{b} , to \mathbf{b}' , where $\mathbf{b}' = \mathbf{b}$ in every component except component 1 where they must differ, so $\mathbf{b}' \neq \mathbf{c}$. Now, we move from \mathbf{b}' to

\mathbf{c}' , where $\mathbf{c}' = \mathbf{b}'$ in every component except component 2 where they must differ, so $\mathbf{c}' \neq \mathbf{b}$. Finally, we move from \mathbf{c}' to \mathbf{c} by once again flipping component 1. Thus, we have extended the length of the path from \mathbf{u} to \mathbf{v} by two. \square

Lemma 5.2.4. ${}_kH_{n+1} \cong Q_n^{2k}$

Proof. Define $\phi: {}_kH_{n+1} \rightarrow Q_n^{2k}$ by $\phi(v) = \phi((b_1, b_2, \dots, b_n, b_{n+1})) = (b_1, b_2, \dots, b_n)$.

We start by showing that ϕ is injective. Let $\mathbf{v}_1, \mathbf{v}_2 \in V({}_kH_{n+1})$ with $\mathbf{v}_1 \neq \mathbf{v}_2$. We know then that $H(\mathbf{v}_1, \mathbf{v}_2) \geq 2$ so $H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2)) \geq 1$ which means $\phi(\mathbf{v}_1) \neq \phi(\mathbf{v}_2)$.

To see that ϕ is surjective we let $\mathbf{v} \in Q_n^{2k}$. Now, we have two cases to consider, first if \mathbf{v} contains an odd number of 1's, then $\mathbf{v}1 \in {}_kH_{n+1}$ and $\phi(\mathbf{v}1) = \mathbf{v}$. Similarly, if \mathbf{v} contains an even number of 1's, then $\mathbf{v}0 \in {}_kH_{n+1}$ and $\phi(\mathbf{v}0) = \mathbf{v}$.

Finally we check that ϕ preserves the edges of the graph.

First, let $\mathbf{v}_1, \mathbf{v}_2 \in {}_kH_{n+1}$ with \mathbf{v}_1 adjacent to \mathbf{v}_2 . Thus, we know that $H(\mathbf{v}_1, \mathbf{v}_2) \leq 2k$. Therefore, $H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2)) \leq 2k$ so in the graph Q_n^{2k} , we must have that $\phi(\mathbf{v}_1)$ is adjacent to $\phi(\mathbf{v}_2)$.

Second, let $\phi(\mathbf{v}_1), \phi(\mathbf{v}_2) \in Q_n^{2k}$ with $\phi(\mathbf{v}_1)$ adjacent to $\phi(\mathbf{v}_2)$. Thus, we know that $H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2)) \leq 2k$. We have two cases to consider. The first case if $H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2))$ is even, then $\phi(\mathbf{v}_1)$ and $\phi(\mathbf{v}_2)$ both must have an even number of 1's which implies that $\phi(\mathbf{v}_1)(n+1) = \phi(\mathbf{v}_2)(n+1) = 0$. Therefore, we have that $H(\mathbf{v}_1, \mathbf{v}_2) = H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2))$.

The second case if $H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2))$ is odd, then without loss of generality we can assume $\phi(\mathbf{v}_1)$ has an even number of 1's and $\phi(\mathbf{v}_2)$ has an odd number of 1's which implies that $\phi(\mathbf{v}_1)(n+1) = 0$ and $\phi(\mathbf{v}_2)(n+1) = 1$. Therefore, we have that $H(\mathbf{v}_1, \mathbf{v}_2) + 1 = H(\phi(\mathbf{v}_1), \phi(\mathbf{v}_2))$. \square

As previously stated in this section we presented a brief background on the injective chromatic number, extended the idea of the injective chromatic number to include paths of any length, and presented a nice relationship between this extended version of the injective chromatic number to the code cover number of error correcting codes. Despite this nice result there remains the open question of finding the injective chromatic number of a 3-regular planar

graph. More specifically, is the injective chromatic number of a 3-regular planar graph 5, 6, or 7?

CHAPTER 6. CONCLUSION

Induced saturation is a robust research area with a wide variety of unexplored problems. The constructions for P_4 -induced saturated graphs that achieve the bound of $\text{indsat}(n, H)$ are rather complex and interesting, bringing to mind the so-called “half graph”. With its applications to satisfiability and Szemerédi’s regularity lemma, it is a very worthwhile problem. This area provides a multitude of open problems. It is of particular interest to find the induced saturation number for C_4 , C_5 , and P_m , for all integers $m \geq 5$. The relationship to satisfiability is particularly fertile ground and we would like to consider interesting and applicable Boolean formulas that are not defined by graph problems. Another generalization of the induced saturation number may be useful for applications, that is, adding a ‘cost’ for black and for white edges. And, of course, there is the remaining open question of finding the injective chromatic number for a 3-regular planar graph.

The proof techniques in finding the induced saturation number for the path on 4 vertices is a nice place to start for potentially finding the induced saturation number of more general paths. The proof yielded a decomposition of the trigraph in which individual components were themselves, induced-saturated. It was rather complicated to get the precise bound of $\lceil \frac{n+1}{3} \rceil$ but it is possible that a more relaxed bound of $n/3 - o(n)$ might have been significantly less detailed. This is the hope for finding the asymptotic values of $\text{indsat}(n, H)$ for more general graphs H . Certainly the result that $\text{indsat}(n, P_4) = \lceil \frac{n+1}{3} \rceil$ was the bulk of the research done and provided valuable insight into the fundamental nature of the induced saturation problem.

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